Tutorial 9. Dispersion of particles.

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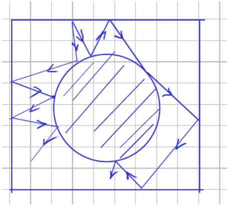
Planetary model of atom

- A key experiment was made by Geiger and Marsden at 1909. They bombarded a gold foil by α-particles and found a dispersion of the α-particles.
- Rutherford established (1911) a formula for effective dispersion of that particles and concluded that mostly mass of atom is condensed in small volume of the atom.

Billiards



Ya.G.Sinai studied
a billiards as a model of kinetic theory of gases. He published work about the billiards and their connection to statistical physics at 1963 in Reports of Soviet Academy of Sciences. At 2014 he award the Abel prize for the works about billiards and statistical physics.

Dispersion of n particles

Let us consider a particle m which will decay on two particles with different masses m_1 and m_2.
The decay can be initialized by pumping of energy ϵ, like egg in the microwaves oven.
Let us consider the particle in a coordinate system with origin on this particle.

- A linear momentum for the decay can written as:

$$m_1\vec{v}_1 + m_2\vec{v}_2 + \cdots + m_n\vec{v}_n = 0.$$

- A kinetic energy after the decay:

$$\mathcal{E} = m_1\frac{v_1^2}{2} + m_2\frac{v_2^2}{2} + \cdots + m_n\frac{v_n^2}{2}.$$

Parameters of decay

These conservation laws allows us to find $v_{1,2}$ for given $m_{1,2}$:

$$\vec{v_2} = -\vec{v_1}\frac{m_1}{m_2}.$$

So a direction for the velocities are opposite to each other. Then we can choose an axis of coordinate which coincides with the direction of $\vec{v_1}$.

$$\epsilon = m_1\frac{v_1^2}{2} + m_2\frac{v_1^2 m_1^2}{2m_2^2}, \quad \epsilon = \frac{m_1}{m_2}(m_1+m_2)\frac{v_1^2}{2}.$$

Then:

$$v_1 = \sqrt{\frac{2m_2\epsilon}{m_1(m_1+m_2)}}, \quad v_2 = -\sqrt{\frac{2m_1\epsilon}{m_2(m_1+m_2)}}.$$

Linear momentum of a system

A linear momentum of n particles:

$$\vec{P} = \sum m_i \vec{v}_i,$$

Define $M = \sum m_i$.
A movement of a center of mass:

$$\vec{v}_c = \frac{\vec{P}}{M}.$$

A linear momentum of system of particles:

$$\sum m_i \vec{v}_i - M\vec{v}_c = 0.$$

Then the linear momentum with respect to the center of mass:

$$\sum_i m_i(\vec{v}_i - \vec{v}_c) = 0.$$

Let us define $\vec{\nu}_i = \vec{v}_i - v_c$, then:

$$\sum m_i \vec{\nu}_i = 0.$$

Kinetic energy

Kinetic energy:

$$E = \sum \frac{m_i}{2} v_i^2.$$

Let us define $\vec{\nu}_i = \vec{v}_i - v_c$

$$E = \sum \frac{m_i}{2} (\nu_i + v_c)^2 = \sum \frac{m_i}{2} \nu_i^2 + \frac{M}{2} v_c^2 + \sum m_i (\vec{\nu}_i, \vec{v}_c).$$

where:

$$\sum m_i (\vec{\nu}_i, \vec{v}_c) = \left(\sum m_i \vec{\nu}_i, \vec{v}_c \right) = (0, \vec{v}_c) = 0.$$

Therefore:

$$E = \mathcal{E} + \frac{M}{2} v_c^2, \quad \mathcal{E} = \sum \frac{m_i}{2} \nu_i^2$$

Collision of two particles

Let us consider a collision of two particles with given m_1, m_2 and $\vec{v}_1$, $\vec{v}_2$.
This means the energy E and $\vec{P}$ are given also.
A velocity of center of mass:

$$\vec{v}_c = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2}.$$

Velocities of the particles with respect to the center of mass before collision are defined as follows:

$$\vec{\nu}_1 = \vec{v}_1 - \vec{v}_c, \quad \vec{\nu}_2 = \vec{v}_2 - \vec{v}_c$$

These vectors are collinear (why?), since:

$$m_1 \vec{\nu}_1 + m_2 \vec{\nu}_2 = m_1(\vec{v}_1 - \vec{v}_c) + m_2(\vec{v}_2 - \vec{v}_c) = \\ m_1 \vec{v}_1 + m_2 \vec{v}_2 - (m_1 + m_2)\vec{v}_c = 0.$$

Collision of two particles

Let us define by $\vec{n}$ an identity vector in a direction of $\vec{\nu}_2$:

$$\vec{n} = \frac{\vec{\nu}_2}{(\vec{\nu}_2, \vec{\nu}_2)} \equiv \frac{\vec{v}_2 - \vec{v}_c}{(\vec{v}_2 - \vec{v}_c, \vec{v}_2 - \vec{v}_c)}.$$

Let us define $\vec{\nu}_i'$ a velocity of i-th particle after collision:

$$m_1 \vec{\nu}_1' + m_2 \vec{\nu}_2' = 0, \quad \vec{\nu}_k' = \nu_k' \vec{n}, \; k = 1, 2.$$

$$\mathcal{E} = \frac{m_1}{2} {\nu_1'}^2 + \frac{m_2}{2} {\nu_2'}^2 + \epsilon.$$

Here ϵ is measure of elasticity for the collision. Denote the full kinetic energy after collision by E':

$$\frac{M}{2} v_c^2 \leq E' \leq E, \quad 0 \leq \epsilon \leq E - \frac{M}{2} v_c^2.$$

Collision of two particles

It yields:

$$\nu_2' = -\frac{m_1}{m_2}\nu_1',$$

$$\mathcal{E} - \epsilon = \frac{m_1}{2}\nu_1'^{\,2} + \frac{m_1^2}{2m_2}\nu_1'^{\,2}.$$

Therefore:

$$\mathcal{E} - \epsilon = \frac{m_1}{2}\nu_1'^{\,2} + \frac{m_1^2}{2m_2}\nu_1'^{\,2}.$$

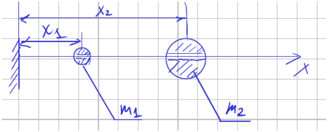
$$\nu_1 = \sqrt{\frac{2(\mathcal{E} - \epsilon)m_2}{m_1(m_1 + m_2)}}, \quad \nu_2 = -\sqrt{\frac{2(\mathcal{E} - \epsilon)m_1}{m_2(m_1 + m_2)}}.$$

Then

$$\vec{v}_1' = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2} + \sqrt{\frac{2(\mathcal{E} - \epsilon)m_2}{m_1(m_1 + m_2)}}\vec{n},$$

$$\vec{v}_2' = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2} - \sqrt{\frac{2(\mathcal{E} - \epsilon)m_1}{m_2(m_1 + m_2)}}\vec{n}.$$

Billiard on half of straight line

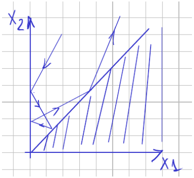


Let us consider two points m_1 and m_2 on a half of line. Position of first point x_1 and position of second one x_2. Suppose that $0 \leq x_1 \leq x_2$. The system can be defined by pair of coordinates these points (x_1, x_2)

A phase space for this system is inner points of angle $\pi/4$ on a plane. Why?

Define velocities before collision like v_1, v_2 and after the collision u_1, u_2.

Changing of variables



Conservation laws for the dynamical system:

$$m_1 v_1 + m_2 v_2 = m_1 u_1 + m_2 u_2,$$

$$\frac{m_1}{2} v_1^2 + \frac{m_2}{2} v_2^2 = \frac{m_1}{2} u_1^2 + \frac{m_2}{2} u_2^2.$$

Let us change variables:

$$\xi_i = \sqrt{m_i} x_i, \quad i = 1, 2.$$

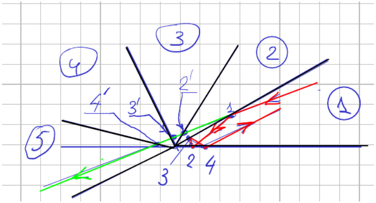
This means the phase space is an angle with the lowest border

$$\frac{\xi_1}{\sqrt{m_1}} = \frac{\xi_2}{\sqrt{m_2}}.$$

The measure of this angle is:

$$\alpha = \arctan\left(\sqrt{\frac{m_1}{m_2}}\right).$$

The conservation laws in new variables

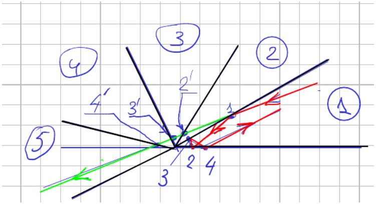


Let's consider the conservation laws for the system in the new variables. The conservation law for the energy looks like

$$\nu_1^2 + \nu_2^2 = \mu_1^2 + \mu_2^2.$$

This means the full speed for the whole system does not change, but, of course, the velocities for both particles change after each collision.

The conservation laws in new variables

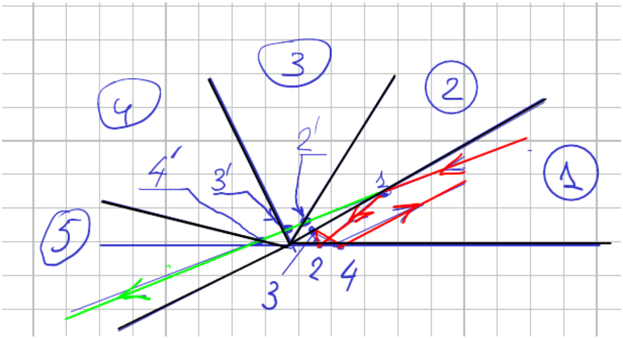


The conservation law for the linear momentum looks like:

$$\sqrt{m_1}\nu_1 + \sqrt{m_2}\nu_2 = \sqrt{m_1}\mu_1 + \sqrt{m_2}\mu_2.$$

This formula shows, that the collision does not change the tangent velocity. (Why?)

The number of collisions



Let's estimate the number of collisions.
Reverse the trajectory after collision. As a result we obtain the number of collisions is:

$$N \leq \left[\frac{\pi}{\alpha}\right] = \left[\frac{\pi}{\arctan\left(\sqrt{\frac{m_1}{m_2}}\right)}\right].$$

The small angle and π

For small angle α we get:

$$\frac{1}{\arctan(\alpha)} = \frac{1}{x} + \frac{x}{3} - \frac{4x^3}{45} + \ldots.$$

Therefore:

$$N \leq \left[\frac{\pi}{\alpha} + \frac{\alpha}{3} + O(\alpha^3)\right],$$

Consider $m_2 \gg m_1$ and let initially $\nu_1 = 0$ and $\nu_2 < 0$.
In this case

$$N\left[\frac{\pi}{\alpha}\right] - 1 = \left[\frac{\pi}{\alpha} + \frac{\alpha}{3} + O(\alpha^3)\right] - 1.$$

If $\sqrt{m_1/m_2} = 10^k$ then

$$N = [10^k \pi] - 1.$$

Dispersion in a central filed

Energy in the central field $U(r)$:

$$E = \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2) + U(r).$$

An angular momentum:

$$M = mr^2\dot{\phi}.$$

The equation for the distance r looks like:

$$\dot{r} = \sqrt{\frac{2}{m}(E - U(r)) - \frac{M^2}{m^2 r^2}}.$$

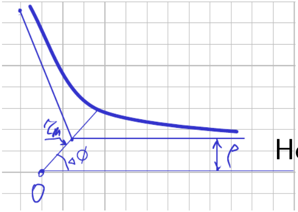
The equation for the angle:

$$d\phi = \frac{M}{mr^2}dt,$$

then:

$$d\phi = \frac{M}{mr^2}\frac{dr}{\sqrt{\frac{2}{m}(E - U(r)) - \frac{M^2}{m^2 r^2}}}.$$

An angle of dispersion



$$\Delta\phi = \int_{r_m}^{\infty} \frac{M}{r^2} \frac{dr}{\sqrt{2m(E - U(r)) - \frac{M^2}{r^2}}}.$$

Here r_m is a root of a following equation:

$$2m(E - U(r_m)) - \frac{M^2}{r_m^2} = 0.$$

On the infinity a linear speed v_0 and a distance between the center line ρ, then the energy and angular momentum are:

$$E = \frac{m}{2} v_0^2, \quad M = m\rho v_0.$$

For the Cologne potential field:

$$U(r) = \frac{c}{r}, \quad c = \mathrm{const}\,.$$

A calculation of $\Delta\phi$

In this case:

$$\int_{r_m}^{\infty} \frac{M}{r^2} \frac{dr}{\sqrt{2m(E - c/r) - \frac{M^2}{r^2}}} = -\int_{r_m}^{\infty} \frac{Md\frac{1}{r}}{\sqrt{2mE - 2m\frac{c}{r} - \frac{M^2}{r^2}}}$$

Let us define $z = 1/r$, then we obtain:

$$\Delta\phi = 2\int_0^{z_m} \frac{dz}{\sqrt{\frac{2mE}{M^2} - \frac{2mc}{M^2}z - z^2}}$$

$$\Delta\phi = 2\int_0^{z_m} \frac{dz}{\sqrt{\frac{2mE}{M^2} + \frac{m^2c^2}{M^4} - (\frac{2mc}{M^2} + z)^2}}$$

Integrating using "Maxima"

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assume(B>0);
assume(A>0);
f:integrate(1/sqrt(A-B*z-z^2),z,0,(B-sqrt(B^2+4*A))/(-2));
f:f,A:2*m*E/M^2,B:2*m*c/M^2;
assume(m>0); assume(v>0); assume(rho>0);
phi:f,M:m*rho*v,E:m*v^2/2,radcan,ratexpand;
tan(phi)^2,radcan;
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As a result we obtain:

$$\tan^2(\Delta\phi) = \frac{m^2 v_0^4 \rho^2}{c^2},$$

This formula defines dependency on the angle andρ. The experiment by Geiger and Marsden at 1909 shows that angles are large for almost all ρ. This means nuclei of the gold foil are rare. This observation allows to understand a structure of nuclei.

Bibliography

- Tabachnikov, Geometry and billiards.
- Landau, Lifshitz, Mechanics.