Tutorial 8. Resonances.

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A pendulum clock. History of invention

Linear oscillator with external force

Parametric resonance in linear oscillator

A pendulum clock

- Clocks as different mechanical and electrical equipment are known at 750 year.
- The first clock was invented in China.
- Watches in towers were known in Europe since 13 century.
- A pendulum as a regulator in the watch was used
- A special spring which worked as a balance was used after Huygens and Hook.
- Now, different periodical physical processes are used as basic oscillators in watches.

Linear oscillator with external force

An equation of linear oscillator looks like:

$$m\ddot{u} + \mu\dot{u} + \omega^2 u = f(t).$$

Here m is mass of the oscillator,
μ is a coefficient of viscous friction, k is a parameter which defines the frequency of the oscillator and $f(t)$ is an external force.

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Here m is mass of the oscillator, μ is a coefficient of viscous friction, k is a parameter which defines the frequency of the oscillator and $f(t)$ is an external force. To get the equation in the simplest form let us divide both part of the equation on ω^2:

$$\frac{du^2}{\frac{\omega^2}{m}dt^2} + \frac{\mu}{\omega\sqrt{m}}\frac{du}{\frac{\omega}{\sqrt{m}}dt} + u = \frac{1}{\omega^2}f(t).$$

An equation for the linear oscillator

More often we will write the linear oscillator in the form

$$u'' + \nu u' + u = h(\tau).$$

Here there is a new independent variable $\tau = \omega t/\sqrt{m}$ and new formulas for the parameters: $\nu = \mu/(\omega\sqrt{m})$, $h(\tau) = f(\Omega\tau)/\omega^2$ and $\Omega = \sqrt{m}/\omega$.

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Any external periodic force can be written in the form of Fourier series:

$$h(\tau) = a_0 + \sum_{k=0}^{\infty} a_k \cos(k\lambda\tau) + b_k \sin(k\lambda\tau)$$

Where $T = 2\pi/\lambda$ is the period of oscillations of the external force and a_k, b_k – Fourier coefficients.

Periodic external force

Therefore the equation for the linear oscillator can be split on series of the equations:

$$u_k'' + \nu u_k' + u_k = a_k \cos(k\lambda\tau),$$

and

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The general solution can be represented as series:

$$u = \sum_{k=0}^{\infty} (u_k(\tau) + v_k(\tau)).$$

Below we assume that all series converge!

Oscillator with single mode without friction.

Let us consider the equation with right-hand side in single form:

$$u_k'' + u_k = a_k \cos(k\lambda t).$$

It is easy to see that the certain solution is:

$$u_k = \frac{a_k}{1 - k^2\lambda^2} \cos(k\lambda t), \quad k^2\lambda^2 \neq 1.$$

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The similar formula can be written for the following equation:

$$v_k'' + v_k = b_k \sin(k\lambda t).$$

It is easy to see that the certain solution is:

$$v_k = \frac{b_k}{1 - k^2\lambda^2} \sin(k\lambda t), \quad k^2\lambda^2 \neq 1.$$

An energy for non-resonant case

For simplicity, let us consider the energy of only one mode of oscillation:

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The energy:

$$E_k = \frac{u_k'^2}{2} + \frac{u_k^2}{2} = \frac{a_k^2}{2(1 - k^2\lambda^2)^2} (k^2\lambda^2 \sin^2(k\lambda\tau) + \cos^2(k\lambda t)).$$

This formula shows that the energy oscillates.

A properties of series for the solution

$$u = \sum k = 0^{\infty} \frac{a_k}{1 - k^2\lambda^2} \cos(k\lambda t) + \frac{b_k}{1 - k^2\lambda^2} \sin(k\lambda t)$$

For the values $1 - k^2\lambda$ close to 0 the term of the solution will have larger value with respect to regular terms.

Resonant case

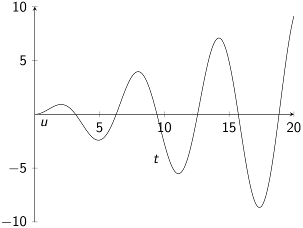


Figure: A resonant curve.

If $k^2\lambda^2 = 1$,
then the equation for this $k^2 = 1/\lambda^2$ looks as follows:

$$u'' + u_k = a_k \cos(\tau).$$

It easy to check that certain solution of this equation is:

$$u_k = \frac{\tau}{2} a_k \sin(\tau).$$

Similar formulas are appropriate for the following equation:

$$v'' + v_k = b_k \sin(\tau)$$

and:

$$v_k = -\frac{\tau}{2} b_k \cos(\tau).$$

An energy for the resonant case

Energy for the resonant case is following:

$$E = \frac{u_k'^2}{2} + \frac{u_k^2}{2} = \frac{a_k^2}{2}(\tau^2 + \sin^2(\tau)).$$

Therefore the energy in the resonant case grows as τ^2.

A forced oscillator with viscous friction

Let us consider an equation with viscous friction:

$$u_k'' + \nu u_k' + u_k = a_k \cos(\omega_k \tau),$$

A certain solution for this equation can be constructed in the following form:

$$u_k = A \cos(\omega_k \tau) + B \sin(\omega_k \tau).$$

Substitute this formula into the equation and equate terms with $\cos(\omega_k \tau)$ and $\sin(\omega \tau)$. As a result we get:

$$-A\omega_k^2 + B\nu\omega_k + A = a_k, \quad -B\omega_k^2 - A\nu\omega_k + B = 0.$$

A solution of this system of equations is:

$$A = \frac{a_k(1-\omega_k^2)}{(1-\omega_k)^2 + \nu^2}, \quad B = \frac{a_k \nu \omega_k}{(1-\omega_k)^2 + \nu^2}.$$

A forced oscillator with viscous friction

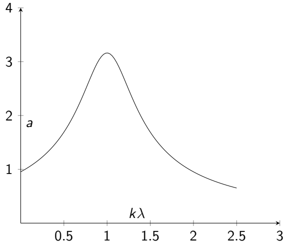


Figure: An amplitude-frequency response.

The
certain solution for the oscillator is

$$u_k = \frac{a_k(1-\omega_k^2)}{(1-\omega_k)^2+\nu^2}\cos(\omega_k\tau) + \frac{a_k\nu\omega_k}{(1-\omega_k)^2+\nu^2}\sin(\omega_k\tau).$$

This
formula can be rewritten in the form:

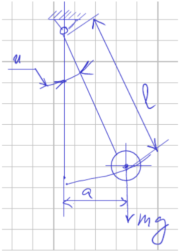
$$u_k = \frac{a_k}{\sqrt{(1-\omega_k)^2+\nu^2}}\cos(\omega_k\tau+\phi_k),$$

$$\phi_k = \arctan\left(\frac{\omega_k\nu}{1-\omega_k^2}\right).$$

Maximum value of energy as $\omega_k = 1$:

$$E = \frac{a_k}{\nu}$$

The pendulum equation



Equation of
a pendulum is derived as a sum of torques:

$$ml^2u'' + mlg\sin(u) = 0$$

where m and l are
the mass and the length of the pendulum
and g is gravitational acceleration.
It is more convenient to rewrite this
equation as follows:

$$u'' + \omega^2\sin(u) = 0, \quad \omega^2 = \frac{g}{l}$$

The parameter ω^2 can be changed by changing the length of the pendulum l. Notice that

$$\omega_1^2 = \frac{g}{l_1}, \quad \omega_2^2 = \frac{g}{l_2}.$$

Therefore:

$$\text{if } \omega_1 < \omega_2, \text{ then } \quad l_1 > l_2.$$

A work of external force

The work of an external force over a period of an oscillation is:

$$A = \int_{\mathcal{L}} f(t)du.$$

Therefore the work of the external force is proportional to the length of the cycle.

The work of the viscous friction:

$$A_v = \int_{\mathcal{L}} u' du.$$

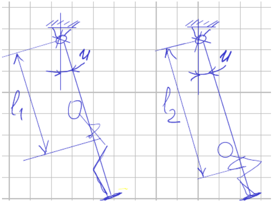
It means, that the force of the viscous friction is proportional to an area inside the cycle.

For the smooth curve the work of the viscous friction grows as a square of the length of the cycle.

Therefore the resonant growth will stop when the work of the external force will be equal to the work of the viscous friction.

But a work of a dry friction is proportional to a length of the cycle and therefore a dry friction cannot stabilize the external force in general case.

A parametric resonance in linear pendulum



Let us consider
a linear part of the pendulum
equation with changed
length of the pendulum:

$$u'' + \omega^2(t)u = 0, \quad \omega(t+2T) = \omega(t),$$

where

$$\omega(t) = \begin{cases} \omega_1, \ (t < T); \\ \omega_2, \ (T < t < 2T). \end{cases}$$

The parameters ω_1 and ω_2 are constants.
General solutions of the equation as $0 < t < T$ has the following form:

$$u_1 = a\cos(\omega_1 t + \phi),$$

The energy for this solution is given by the formula:

$$E = \left(\frac{(u_1')^2}{2} + \omega_1^2 \frac{u_1^2}{2}\right) m l_1^2 = m l_1^2 \omega_1^2 \frac{a^2}{2} = m g l_1 \frac{a^2}{2}.$$

Parametric resonance in linear oscillator

On the next interval of time $T < t < 2T$ the appropriate form of the general solution looks like:

$$u_2 = b_1 \cos(\omega_2(t - T)) + b_2 \sin(\omega_2(t - T)).$$

An equivalent form of of this solution is:

$$u_2 = b \cos(\omega_2(t - T) + \phi_2), \quad b = \sqrt{b_1^2 + b_2^2}, \quad \tan(\phi_2) = -\frac{b_2}{b_1}.$$

Initial conditions for this solution look like:

$$u_2|_{t=T} = b_1, \quad u_2'|_{t=T} = \omega_2 b_2.$$

To prolong the solution on this interval one should match this form of the solution with previous one:

$$b_1 = a \cos(\omega_1 T + \phi), \quad b_2 = -\frac{\omega_1}{\omega_2} a \sin(\omega_1 T + \phi).$$

The formula for the full mechanical energy in this case looks like:

$$E = \left(\frac{u_2'^2}{2} + \omega_2^2 \frac{u_2^2}{2}\right) m l_2^2 = mgl_2 \frac{b_1^2 + b_2^2}{2}.$$

Changing of the energy

$$E_2 = mgl_2 \frac{a^2}{2} \left(\cos^2(\alpha) + \frac{l_2}{l_1} \sin^2(\alpha) \right), \quad \alpha = \omega_1 T + \phi.$$

The previous value of the energy:

$$E_1 = mgl_1 \frac{a^2}{2}.$$

Therefore:

$$E_2 = \frac{l_2}{l_1} \left(\cos^2(\alpha) + \frac{l_2}{l_1} \sin^2(\alpha) \right) E_1.$$

If $l_2 > l_1$, then $E_2 > E_1$.
In the parametric resonance the energy grows exponentially.