Tutorial 7. Potential energy, equilibrium and Lyapunov stability.

O.M. Kiselev
o.kiselev@innopolis.ru

Innopolis university

The law of conservation energy

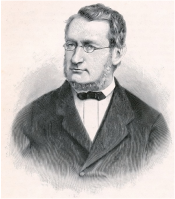
 

Figure: James Prescott Joule and Julius Robert von Mayer

The most famous antique books about physics are called Metaphysics by Aristotle.
Aristotle used the two different quantities which define the Energy. Namely there was potential energy and actual energy.

A discussion about the conservation law of energy was passed at middle of 19-th century between von Mayer and Joule. They argue priority on the invention of equivalent between a heat and mechanical energy. This discussion involves a lot of famous physician of that time like Thomson and Helmholtz.

Potential Energy

Let us consider formula for potential energy $U(x)$.
A derivative of the potential energy on x defines the conservative force:

$$F(x) = \frac{\partial U(x)}{\partial x}.$$

- The simplest example is a potential energy depends linearly on an independent variable like a following form:

$$U(x) = mgx, \quad F \equiv mg.$$

- A quadratic form of dependency on x:

$$U(x) = k\frac{x^2}{2}, \quad F(x) \equiv x.$$

- Central field like a gravitational field and Coloumb field:

$$U(x) = -\frac{Gm}{x}, \quad F(x) \equiv -\frac{Gm}{x^2}.$$

Potential energy and equilibrium

The equilibrium means

$$F(x) \equiv 0, \quad \text{or} \quad \frac{\partial U(x)}{\partial x} \equiv 0.$$

If the potential energy is smooth function, then near the equilibrium

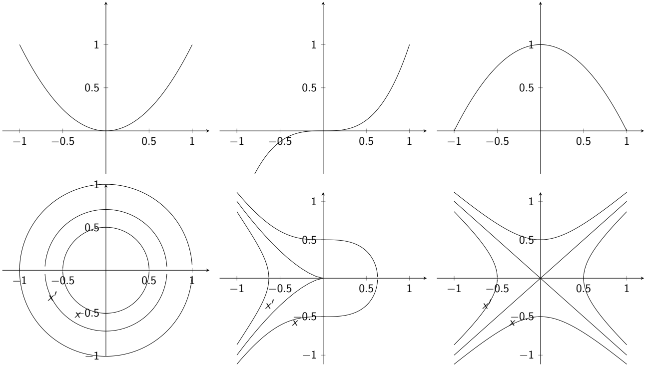
$$\frac{\partial U(x)}{\partial x} \equiv F(x_0) = 0.$$

Potential energy can be represented by a segment of Taylor series:

$$U(x) = U(x_0) + \frac{1}{2}U''(x_0)(x - x_0)^2 + \frac{1}{3!}U'''(x_0 + \theta(x, x_0))(x - x_0)^3,$$

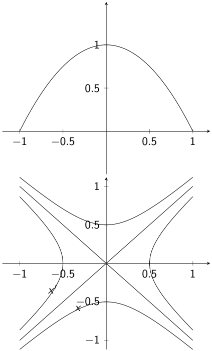
here $\theta \in (x_0, x)$.

Different types of equilibrium



- If $U''(x_0) > 0$ then point $x = x_0$ is a minima of the potential energy.
- If $U''(x_0) = 0$ then point $x = x_0$ is an inflection point.
- If $U''(x_0) < 0$ then point $x = x_0$ is a maxima of the potential energy.

Saddle point



This type
of the equilibrium is called a saddle.
Typical dynamical equation for this
case of the potential energy looks as:

$$m\ddot{x} - \lambda^2 x = 0.$$

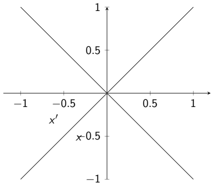
Here
the coefficient λ^2 is written as some
square to show the negative value
for the term $-\lambda^2 x$ in the equation.
A general solution for this equation is
given by the formula:

$$x = C_1 e^{\lambda t/\sqrt{m}} + C_2 e^{-\lambda t/\sqrt{m}}, \quad C_1, C_2 \in \mathbb{R}.$$

The full mechanical energy is:

$$E = m\frac{\dot{x}^2}{2} - \lambda^2 \frac{x^2}{2}.$$

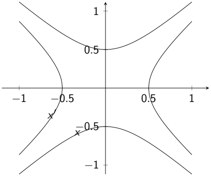
Saddle trajectories



If $E = 0$ then
we have five different trajectories.

- $(0, 0)$ is the saddle point.
- $\dot{x} = (\lambda/\sqrt{m})x$, as $x > 0$.
  - A parametric formula for this line looks like $x = e^{\lambda t/\sqrt{m}}$ and $\dot{x} = (\lambda t/\sqrt{m})e^{\lambda t/\sqrt{m}}$.
- $\dot{x} = (\lambda/\sqrt{m})x$, as $x < 0$.
  - A parametric formula for this line looks like $x = -e^{\lambda t/\sqrt{m}}$ and $\dot{x} = -(\lambda t/\sqrt{m})e^{\lambda t/\sqrt{m}}$.
- $\dot{x} = -(\lambda/\sqrt{m})x$, as $x > 0$.
  - A parametric formula for this line looks like $x = e^{-\lambda t/\sqrt{m}}$ and $\dot{x} = -(\lambda t/\sqrt{m})e^{-\lambda t/\sqrt{m}}$.
- $\dot{x} = -(\lambda/\sqrt{m})x$, as $x < 0$.
  - A parametric formula for this line looks like $x = -e^{-\lambda t/\sqrt{m}}$ and $\dot{x} = (\lambda t/\sqrt{m})e^{-\lambda t/\sqrt{m}}$.

Trajectories near the saddle: $E = -2C_1C_2\lambda^2$.



For regular trajectories: If $E > 0$ then we have two different trajectories for given E.

$$\dot{x} = \pm\frac{1}{\sqrt{m}}\sqrt{2E + \lambda^2 x^2}.$$

A parametric form for that curves is looking as follows:

$$x = C_1 e^{\lambda t/\sqrt{m}} + C_2 e^{-\lambda t/\sqrt{m}},$$
$$\dot{x} = C_1(\lambda/\sqrt{m})e^{\lambda t/\sqrt{m}} - C_2(\lambda/\sqrt{m})e^{-\lambda t/\sqrt{m}}.$$

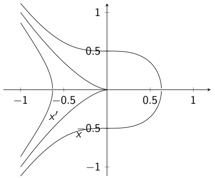
If $E < 0$ then we have two different trajectories for given E.

$$x = \pm\frac{1}{\lambda}\sqrt{-E + m\dot{x}^2},$$

A parametric form for that curves is looking as follows:

$$x = C_1 e^{\lambda t/\sqrt{m}} + C_2 e^{-\lambda t/\sqrt{m}},$$
$$\dot{x} = C_1(\lambda/\sqrt{m})e^{\lambda t/\sqrt{m}} - C_2(\lambda/\sqrt{m})e^{-\lambda t/\sqrt{m}},$$

Special case $U''(x_0) = 0$



Typical equation
for this special case looks like

$$m\frac{\dot{x}^2}{2} + kx^3 = E.$$

If $E = 0$
we have three different trajectories.

- $(0,0)$ is a saddle-center point.
- $\dot{x} = \sqrt{\frac{k}{m}}\sqrt{-2x^3}, \quad x < 0.$
- $\dot{x} = -\sqrt{\frac{k}{m}}\sqrt{-2x^3}, \quad x < 0.$

If $E \neq 0$ then there exist one trajectory for given value E, this trajectory tends from $\dot{x} \to -\infty$ as $t \to -\infty$ and $\dot{x} \to -\infty$ as $t \to \infty$.

$$\dot{x} = \sqrt{\frac{k}{m}}\sqrt{2E - 2x^3}, \quad E > x^3.$$

Mechanical energy and friction

A friction is one of cause of decreasing energy in mechanical systems.
Let us consider a movement form one point x_0 at the moment $t = t_0$ to the point x at the moment t.
The difference between a full mechanical energy can be written in the form:

$$\Delta E = E(t) - E(t_0).$$

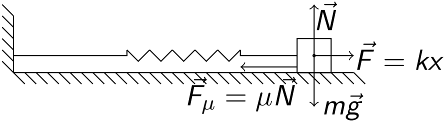
A work of the friction can be written as an integral over the path:

$$A = \int_{x_0}^{x(t)} F d\xi.$$

A consequence of a theorem of changing mechanical energy is the following formula:

$$E(t) - E(t_0) = A.$$

A dry friction in the oscillations



Let us consider the movement with dry friction in a following mechanical system as shown in the figure. A full mechanical energy of the system has a following form:

$$E = \frac{m\dot{x}^2}{2} + k\frac{x^2}{2}.$$

A dry friction in the oscillations

A changing of this energy depends on the direction of the movement:

$$\dot{E} = m\dot{x}\ddot{x} + k\dot{x}u = (m\ddot{x} + kx)\dot{x}.$$

We know a differential law for the movement:

$$m\ddot{x} + \mu mg\,\mathrm{sign}(\dot{x}) + kx = 0$$

Using this equation we obtain:

$$\dot{E} = -\mu mg\,\mathrm{sign}(\dot{x})\dot{x}.$$

The right-hand side of this formula is non-positive. Then the value of the energy decreases.

A viscous friction in the oscillator

The same oscillator for viscous friction

$$F = -\mu m g \dot{x}.$$

In this case the equation for the oscillator looks like

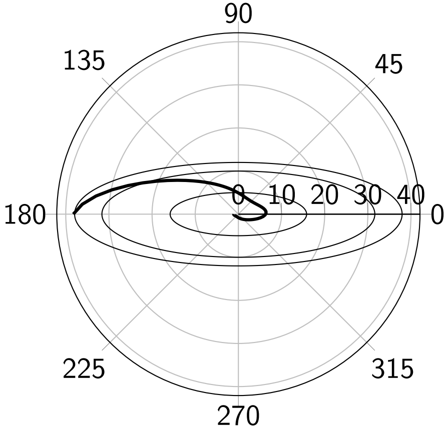
$$m\ddot{x} + kx = -\mu m g \dot{x}.$$

Therefore the energy for the oscillator with viscous friction:

$$\dot{E} = (m\ddot{x} + kx)\dot{x} = -\mu m g \dot{x}^2.$$

So the energy for the oscillator with viscous friction decreases.

Trajectories with decreasing energy



Curves depend on the energy looks like the ellipses. Therefore the trajectories with decreasing energy crosses through such ellipses. The angle between the tangent to the ellipse and the trajectories:

$$\tan(\alpha) = -\mu m g \dot{x}^2.$$

The Lyapunov function and stability of the equilibrium

Definition

The trivial solution $x \equiv 0$ is stable if $\forall \varepsilon > 0$ and $\forall t_0$ $\exists \delta(\varepsilon, t_0) > 0$ and $\exists x_0$ $|x_0| < \delta \leq \varepsilon$ then

$$|x(t)| < \varepsilon, \quad \forall t > t_0.$$

Theorem (Second Lyapunov's theorem)

If $\exists L(x)$ as $|x| < \varepsilon$ and

- $L(x) = 0$ *if and only if* $x = 0$;
- $L(x) > 0$ *if and only if* $x \neq 0$;
- $\dot{L}(x) \leq 0$ *for all* $x < \epsilon$ *and* $t > t_0$,

Then the solution is stable.

It easy to see that the formula for the full energy is the Lyapunov function for the linear oscillator with friction.

Non-linear oscillator with friction

Let us consider a non-linear oscillator with viscous friction:

$$\ddot{x} + \mu\dot{x} + f(x) = 0, \quad f(0) \equiv 0, \quad f'(0) \equiv k^2 > 0.$$

A full energy for this oscillator is:

$$\frac{\dot{x}^2}{2} + F(x) = E, \quad F'(x) \equiv f(x).$$

The condition $f'(0) > 0$ means that the potential energy has minima in the point $x = 0$ and:

$$\dot{E} \equiv \dot{x}(\ddot{x} + f(x)) = -\mu\dot{x}^2.$$

Non-linear oscillator with friction

In small neighbourhood of $x = 0$ we can use a segment of a Taylor series for the $F(x)$:

$$F(x) = k^2x^2 + O(x^3), \quad |x| < \varepsilon.$$

Let us consider $\vec{x} = (x, \dot{x})$ and there exist $\varepsilon > 0$ such that for $|\vec{x}| < \varepsilon$

- $E(\vec{x}) = 0$ if and only if $\vec{x} \equiv 0$;
- $E(\vec{x}) > 0$ if and only if $\vec{x} \not\equiv 0$;

Therefore E is a Lyapunov function for the non-linear oscillator.

Summary