Tutorial 7. Filters

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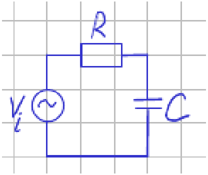
First order DE for circuits

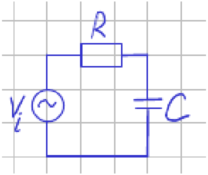


The simplest form
of AC circuit contains both resistance
and capacitance and a voltage source.
The voltage
on the resistor can be written as

$$v_r = ir, \quad i = i(t), \quad r \in \mathbb{R}.$$

Here $i(t)$ is complex valued function and r is a real valued parameter, which we consider as a value of resistance.

Assumptions



The voltage on the capacitor
is considered as a following formula

$$v_c = \frac{1}{C}\int_{t_0}^{t} i(\tau)d\tau,$$

here $C \in \mathbb{R}$ is real
valued parameter modeled of capacity.

For simplicity we assume $t_0 = 0$.

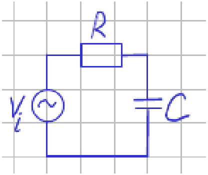
Assumptions

We suppose that the source generates the alternating voltage which are:

$$v(t) = V_i e^{j\omega t}, \quad j^2 = -1, \quad V_i \in \mathbf{R}, \quad \omega \in \mathbb{R}.$$

Here V_i is an amplitude of external voltage, which is a real valued external parameter. The value ω is a frequency of the alternating source.

First order DE for circuits



The current for
all circuit is the same value because the circuit has only one mesh without any threads. Therefore the Kirchhoff law for the current has a trivial form:

$$i = i(t).$$

The Kirchhoff law for the voltage over the mesh can be written in following form:

$$v_r + v_c = v,$$

or more detailed form is following:

$$ir + \frac{1}{C}\int_{t_0}^{t} i(\tau)d\tau = V_i e^{j\omega t}.$$

After differentiating one gets:

$$i' r + \frac{1}{C} i = j\omega V_i\, e^{j\omega t}.$$

Cauchy problem

For the solution of the following equation

$$i' r + \frac{1}{C} i = j\omega V_i \, e^{j\omega t}$$

we assume an initial condition:

$$i\big|_{t=0} = 0.$$

A general solution of this Cauchy problem can be written as follows:

$$i(t) = x(t) + i_c(t),$$

where $x(t)$ is a solution of joint homogeneous equation and $i_c(t)$ is a certain solution.

Certain solution

Let us consider a certain solution:

$$i_c(t) = A\,e^{j\omega t}, \quad A \in \mathbb{C}.$$

Substitute the formula into the equation. It yields:

$$j\omega r A\,e^{j\omega t} + \frac{1}{C} A\,e^{j\omega t} = j\omega V_i\,e^{j\omega t}.$$

Then

$$A(j\omega rC + 1) = j\omega CV_i, \quad A = \frac{j\omega CV_i}{j\omega rC + 1}.$$

So

$$i_c(t) = \frac{j\omega CV_i}{j\omega rC + 1}\,e^{j\omega t}.$$

General solution of the DE

General solution has a form of sum of certain solution and general solution of the joint equation:

$$x' r + \frac{1}{C} x = 0, \quad x = x_0 \, e^{-t/(rC)}.$$

Therefore:

$$i(t) = x_0 \, e^{-t/(rC)} + \frac{j\omega C V_i}{j\omega rC + 1} \, e^{j\omega t}.$$

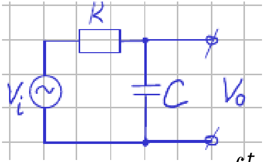
The value of x_0 one can obtain from the initial condition:

$$x_0 + \frac{j\omega C V_i}{j\omega rC + 1} = 0, \quad x_0 = -\frac{j\omega C V_i}{j\omega rC + 1}.$$

First term tends to 0 as $t \to \infty$ therefore the solution stabilizes to

$$i(t) \sim \frac{j\omega C V_i}{j\omega rC + 1} \, e^{j\omega t}, \quad t \gg 1.$$

RC-filter



Let us consider
the voltage on the capacitor.

$$v_0 = \frac{1}{C}\int_0^t i(\tau)d\tau,$$

$$v_0(t) = \frac{x_0}{C}\int_0^t e^{-\tau/(rC)}d\tau + \frac{1}{C}\frac{j\omega CV_i}{j\omega rC+1}\int_0^t e^{j\omega\tau}d\tau =$$

$$-\frac{j\omega rCV_i}{j\omega rC+1}(-e^{-t/(rC)}+1) + \frac{V_i}{j\omega rC+1}(e^{j\omega t}-1).$$

Then the alternating part of the voltage is follows:

$$v_0 \sim \frac{1}{j\omega rC+1}V_i\, e^{j\omega t}, \quad t \gg 1.$$

RC-filter

In this case the RC circuit we can consider as a filter.
So for an amplitude v_i for input voltage one obtains the amplitude v_0 for output voltage. The fraction

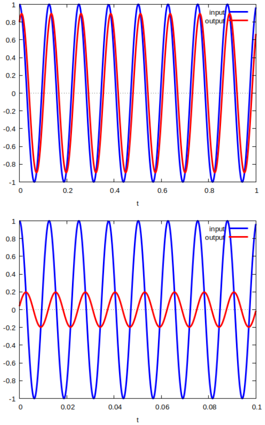
$$H(j\omega) = \frac{V_0}{V_i} = \frac{1}{j\omega rC + 1}.$$

In the exponential form one gets

$$H(j\omega) = \frac{\sqrt{\omega^2 r^2 C^2 + 1}}{\omega^2 r^2 C^2 + 1} e^{i\phi} = \frac{e^{i\phi}}{\sqrt{\omega^2 r^2 C^2 + 1}}$$

$$\tan\phi = -\omega rC.$$

Low-pass filter

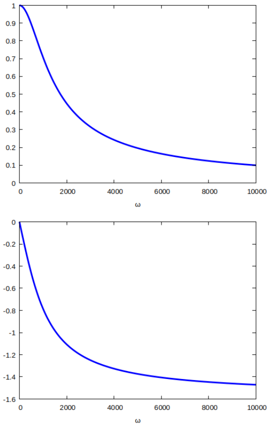


The final formula for the magnitude of the fraction is traditionally written in the following form:

$$|H(j\omega)| = \frac{1}{\sqrt{\left(\frac{\omega}{\omega_0}\right)^2 + 1}}, \quad \omega_0 = \frac{1}{rC}.$$

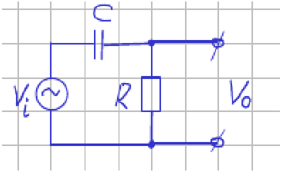
Here ω_0 is called *cutoff frequency*.

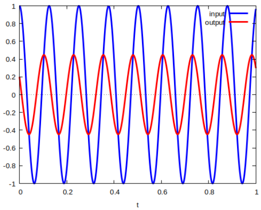
Low-pass filter

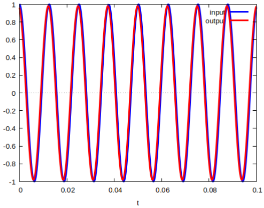


If $\omega \to \infty$ then $|H(j\omega)| \to 0$.
Otherwise if $\omega = 0$, then $|H(j\omega)| = 1$.
Therefore the *RC*-filter is the filter
for high frequencies, or *low-pass filter*.
If $\omega \to \infty$ then $\tan(\phi) = -\omega rC \to \infty$
therefore $\phi \to -\pi/2$.
If $\omega \to 0$ then
$\tan(\phi) = -\omega rC \to 0$ therefore $\phi \to 0$.

High-pass filter







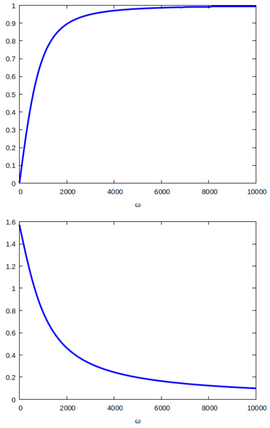
Let us consider
the voltage on the resistor:

$$v_0 = ri(t) = r\frac{j\omega C V_i}{j\omega rC + 1}e^{j\omega t}.$$

Therefore the fraction between input and output voltage is

$$H(j\omega) = \frac{j\omega Cr}{j\omega rC + 1}.$$

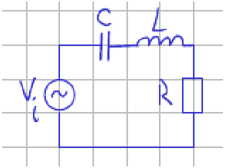
High-pass filter



$$H(j\omega) = \frac{j\omega Cr}{j\omega rC + 1}.$$

If $\omega \to \infty$ then $|H(j\omega)| \to 1$.
Otherwise,
if $\omega = 0$, then $|H(j\omega)| = 0$.
Therefore the such filter is the filter for low frequencies, or *high-pass filter*.
If $\omega \to \infty$
then $\tan(\phi) \to 0$ therefore $\phi \to 0$.
If $\omega \to 0$
then $\tan(\phi) \to \infty$ therefore $\phi \to \pi/2$.

Circuit with inductor



Let us consider the circuit with inductor. This for such circuit the Kirchhoff voltage law looks like the following equation:

$$Li' + ri + \frac{1}{C}\int_{t_0}^{t} i(\tau)d\tau = v_i(t).$$

Differentiate this equation:

$$Li'' + ri' + \frac{1}{C}i = j\omega V_i e^{j\omega t}.$$

A general solution for the equation can be written as a sum of a certain solution and a general solution of the joint equation:

$$Lx'' + rx' + \frac{1}{C}x = 0.$$

Solution of the joint equation

The solution for the joint equation:

$$x = B_1 e^{\lambda_1 t} + B_2 e^{\lambda_2 t},$$

where $\lambda_{1,2}$ are solution of the quadratic equation:

$$L\lambda^2 + r\lambda + \frac{1}{C} = 0.$$

Therefore

$$\lambda_1 = \frac{-r + \sqrt{r^2 - 4LC}}{2LC}, \quad \lambda_2 = \frac{-r - \sqrt{r^2 - 4LC}}{2LC}.$$

It it easy to see both λ are negative. Therefore

$$x \to 0, \quad t \to \infty.$$

A certain solution

A certain solution of the equation has a form:

$$i_0(t) = A\, e^{j\omega t},$$

Then:

$$\left(-L\omega^2 + jr\omega + \frac{1}{C}\right) A e^{j\omega t} = j\omega V_i e^{j\omega t}.$$

It yields:

$$A = \frac{j\omega C}{1 - LC\omega^2 + jrC\omega} V_i$$

General solution

Therefore the general solution has a form:

$$i(t) = B_1 e^{\lambda_1 t} + B_2 e^{\lambda_2 t} + \frac{j\omega C}{1 - LC + jrC\omega} V_i e^{j\omega t}.$$

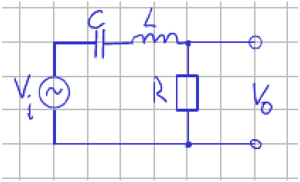
As $t \to \infty$ one gets:

$$i(t) \sim \frac{j\omega C}{1 - LC\omega^2 + jrC\omega} V_i e^{j\omega t}.$$

The voltage on the resistor is

$$v_0 = ir \sim \frac{j\omega rC}{1 - LC\omega^2 + jrC\omega} V_i e^{j\omega t}$$

Bandpass filter



Therefore

$$H(j\omega) = \frac{j\omega rC}{1 - LC\omega^2 + jrC\omega} = \frac{j\omega rC}{1 - LC\omega^2 + jrC\omega}$$

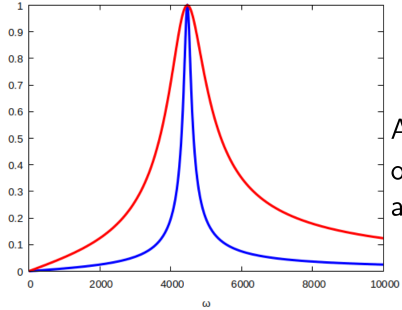
Often this
formula is written as follows

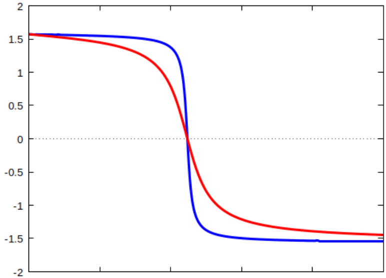
$$H(j\omega) = \frac{j\omega rC}{1 - LC\omega^2 + jrC\omega} = \frac{j\omega rC}{\left(1 + j\frac{\omega}{\omega_1}\right)\left(1 + j\frac{\omega}{\omega_2}\right)}$$

Here

$$\frac{1}{\omega_1\omega_2} = LC, \quad \frac{1}{\omega_1} + \frac{1}{\omega_2} = rC.$$

Resonance and bandwidth





Analyze the ratio formula:

$$H(j\omega) = \frac{j\omega rC}{1 - LC\omega^2 + jrC\omega}.$$

As $\omega_* = \frac{1}{\sqrt{LC}}$ the magnitude of the ration has a maximum and phase shift is zero:

$$H\left(j\frac{1}{\sqrt{LC}}\right) = 1, \quad \phi = 0.$$

The value $rC\omega$ defines the sharpness of the resonant peak.

$$Q = \frac{1}{rC\omega_*} = \frac{1}{r}\sqrt{\frac{L}{C}}$$

is called *quality factor*.

Summary

- Derivation of ordinary differential equations for circuits.
- A low-pass RC-filter.
- A high-pass RC-filter.
- Band-pass filter