Tutorial 6. Energy and Newton's laws.

O.M. Kiselev
o.kiselev@innopolis.ru

Innopolis university

Energy and Newton's laws
History of invention
Second Newton's law and phase plane

From *vis viva* to *kinetic energy*

An understanding of mechanical properties which allows a free movement of a material body was coming from antique. There were clear that property is defined by following characteristics:

- an inertial mass;
- speed of the motion.

But the formal attitude of this property was obtained only by Daniel Bernoulli in his work at 1741 year.
Moreover till works by Coriolis and Poncelet (1839 year) (during a century!) there was missunderstanding between:

- *linear momentum* as $m\vec{v}$,
- *vis viva* as mv^2
- and *kinetic energy* as $mv^2/2$.

The mechanical work and formula for the kinetic energy

In the 19-th century the formula for the work of steam machine looks like:

$$A(x) = \int_0^x F(\xi) d\xi,$$

where F – force which can be depended on a position of the piston and x is a piston stroke (changing of piston position under the pressure of the steam).
So now one can consider the second Newton's law in the form:

$$m\ddot{x} = F(x).$$

The mechanical work and formula for the kinetic energy

Then multiply both parts on the velocity $\dot{x}$:

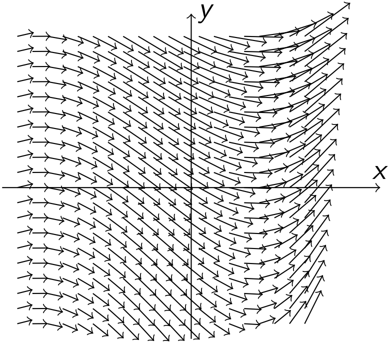
$$m\ddot{x}\dot{x} = F(x, \dot{x}, t)\dot{x}$$

and integrate over t:

$$m\frac{\dot{x}^2}{2} - m\frac{\dot{x_0}^2}{2} = \int_0^t F(x(\tau), \dot{x}(\tau), \tau)d\tau.$$

So we obtain the law for changing kinetic energy but also we see that the kinetic energy as $m\dot{x}^2/2$ have straightforward connection with mechanical work. Such formula appears for example in a work of Coriolis, "Du calcul de l'effet des machines", 1829 (see page 15 of that work).

Second Newton's law and phase plane



Let us consider second Newton's for one-dimensional movement with special form of the force. Namely for the force did not dependent on time:

$$m\ddot{x} = f(x, \dot{x}).$$

This equation of the second order can be rewritten into two firs order equation:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \frac{1}{m} f(x, y). \end{aligned}$$

The solution of this system can be considered as a parametrically defined curve $(x(t), y(t))$ on the plane (x, y).

Typical phase spaces

- A configuration space for the pendulum is $\mathbb{S}$ and phase space for pendulum $(\alpha, \dot{\alpha}) \in \mathbb{S} \times \mathbb{R}$.
- A configuration space for spherical pendulum is $\mathbb{S} \times [-\pi/2, \pi/2]$ and a phase space for the is $(\phi, \psi, \dot{\phi}, \dot{\psi}) \in \mathbb{S} \times [-\pi/2, \pi/2] \times \mathbb{R}^2$.
- A configuration space for a double pendulum is $\mathbb{S} \times \mathbb{S} = \mathbb{T}$ and the phase space for the is $\mathbb{T} \times \mathbb{R}^2$.

Trajectory and full energy

The formula for full energy connects the kinetic energy and potential energy.

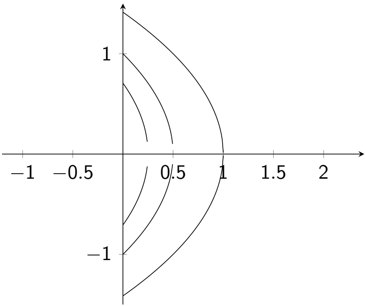
$$\frac{m\dot{x}^2}{2} + \Pi(x) = E.$$

This formula give the dependency of $\dot{x}$ on $\Pi(x)$ like

$$\dot{x} = \pm\sqrt{\frac{2}{m}(E - \Pi(x))}.$$

Thus for one dimensional movement one can obtain the trajectory without of solution for the differential equation.

Trajectories and full energy



Consider a free fall. The Newton's second law:

$$m\ddot{x} = -mg.$$

Multiply on $\dot{x}$ both part of the formula. One time integration over t yields:

$$m\frac{\dot{x}^2}{2} = -mgx + E.$$

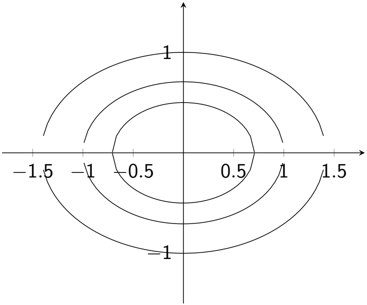
The trajectories are parabolas:

$$\dot{x}^2 = \frac{2}{m}E - 2gx.$$

Then the trajectories on the phase plane with energy E is defined by the formula:

$$\dot{x} = \pm\sqrt{\frac{2}{m}E - 2gx}$$

Trajectories and full energy



If the movement
is defined by the Hook's law:

$$m\ddot{x} = -kx,$$

then after multiplying
both part of the equation on $\dot{x}$:

$$m\ddot{x}\dot{x} = -kx\dot{x},$$

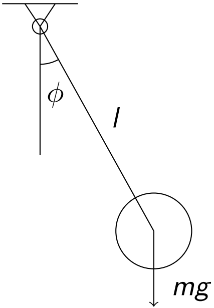
and integration over t we will get the following formula for the full energy:

$$m\frac{\dot{x}^2}{2} = -k\frac{x^2}{2} + E.$$

This means the trajectories are ellipses:

$$k\frac{x^2}{2} + \frac{\dot{x}^2}{2} = E.$$

A pendulum



$$ml^2\ddot{\phi} + gml\sin(\phi) = 0$$

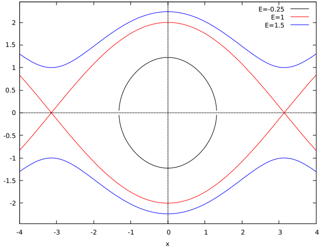
$$\ddot{\phi} + \frac{g}{l}\sin(\phi) = 0.$$

$$\frac{d^2}{\frac{g}{l}dt^2}\phi + \sin(\phi) = 0.$$

$$\tau = \sqrt{\frac{g}{l}}\,t.$$

$$\frac{d^2\phi}{d\tau^2} + \sin(\phi) = 0.$$

Energy of the pendulum



$$\dot{\phi}\frac{d^2\phi}{d\tau^2} + \sin(\phi)\dot{\phi} = 0.$$

$$\frac{\dot{\phi}^2}{2} - \cos(\phi) = E.$$

$$\dot{\phi} = \pm\sqrt{2E + 2\cos(\phi)}.$$

- Oscillation movement is defined by values of the energy $-1 < E < 1$.
- A special case of the energy $E = 1$ defines the separatrix motion.
- A rotation is defined by the values of the energy $E > 1$.

Trajectories and full energy

Let us consider the movement in the central gravitational force. The sum of the radial forces are

$$m\ddot{r} = mr\dot{\phi}^2 - G\frac{Mm}{r^2}.$$

The equation for the tangential forces:

$$m\frac{d}{dt}(r^2\dot{\phi}) = 0.$$

Every one can recall our tutorial about Newton's gravitational law that after rewriting we obtained the following system of equations:

$$\ddot{\rho} = \frac{1}{\rho^3} - \frac{1}{\rho^2},$$

and

$$\rho^2\dot{\phi} = 1.$$

Integration of the equation of motion

Multiply the equation for the radial motion on $\dot{\rho}$:

$$\dot{\rho}\ddot{\rho} = \frac{\dot{\rho}}{\rho^3} - \frac{\dot{\rho}}{\rho^2}.$$

It yields:

$$\frac{d}{d\tau}\left(\frac{\dot{\rho}^2}{2}\right) = -\frac{1}{2}\frac{d}{d\tau}\left(\frac{1}{\rho^2}\right) + \frac{d}{d\tau}\left(\frac{1}{\rho}\right).$$

Or the same form:

$$\frac{d}{d\tau}\left(\frac{\dot{\rho}^2}{2}\right) = \frac{d}{d\tau}\left(-\frac{1}{2}\frac{1}{\rho^2} + \frac{1}{\rho}\right).$$

Integrate the left and right-hand sides of the equation:

$$\frac{\dot{\rho}^2}{2} = -\frac{1}{2}\frac{1}{\rho^2} + \frac{1}{\rho} + E.$$

Energy levels for different type of motions

Now we obtain two first order equations with additional constant of integration E:

$$\rho^2\dot{\phi} = 1$$

and

$$\frac{\dot{\rho}^2}{2} + \frac{1}{2}\frac{1}{\rho^2} - \frac{1}{\rho} = E, \quad E = \text{const}\,.$$

The different values of the energy defines the different types of movement.

The fit values for E:

$$E \geq \frac{1}{2}\frac{1}{\rho^2} - \frac{1}{\rho},$$

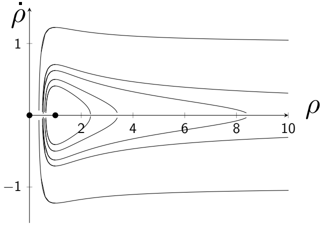
Then minimal value of E can be obtained from the equation:

$$\frac{dE}{d\rho} \equiv \frac{1}{\rho^2} - \frac{1}{\rho^3} = 0.$$

Then

$$E \geq -1/2.$$

Energy levels for different type of motions



Let us consider the numbers of the real roots for the equation

$$\frac{1}{2}\frac{1}{\rho^2} - \frac{1}{\rho} - E = 0.$$

The same for $\rho \neq 0$:

$$2E\rho^2 + 2\rho - 1 = 0.$$

The determinant of this quadratic equation is:

$$D = 4 + 8E.$$

We are interested in real valued solutions then we obtain the same condition $E \geq -1/2$.

Different types of movement

The formula from Tutorial 4 looks like:

$$\rho = \frac{1}{1 + \sqrt{1 + 2E}\sin(\phi)}.$$

- If $E = -1/2$ then $\rho = 1$, $\dot{\rho} = 0$. The trajectory is a circle.
- If $-1/2 < E < 0$ then we obtain two positive solutions of the quadratic equation and the trajectory is an ellipse:

$$\rho_{1,2} = \frac{-2 \pm \sqrt{4 + 8E}}{4E}.$$

- If $E = 0$ then

$$\frac{1}{2}\frac{1}{\rho^2} - \frac{1}{\rho} = 0, \quad \rho = 1/2.$$

  Then $\dot{\rho} = 0$ as $\rho = 1/2$ or $\rho = \infty$. This trajectory is a parabola.
- If $E > 0$ then there exists only one positive value when $\dot{\rho} = 0$:

$$\rho = \frac{-2 + \sqrt{4 + 8E}}{4E}.$$

Orbital speed and escape velocity

- The orbital velocity can be obtained as follows

$$\rho = 1, \quad \dot{\phi} = 1, \quad v_1 = \rho\dot{\phi} = 1.$$

- The escape velocity equals to:

$$\rho = 1/2, \quad \dot{\phi} = 1/\rho^2 = 4, \quad v_2 = \rho\dot{\phi} = 2.$$