Tutorial 6. AC and complex valued approach

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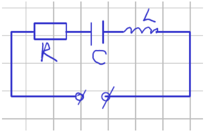
Properties of voltage on elements of circuit

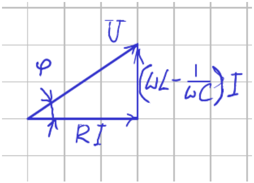


Let us consider serial connection
of resistance, capacitor and inductor.
In this case for all elements
of the circuit the current the same and

$$I = I_0 \cos(\omega t).$$

- The voltage on the resistance: $V_R = I_0 R \cos(\omega t)$. The direction coincides to the current.
- the voltage on the capacitor: $V_c = \frac{1}{C} \int_0^t I(t) dt = \frac{I_0}{C} \int_0^t cos(\omega t) dt = \frac{I_0}{C\omega} \sin(\omega t) = R_C I_0 \cos\left(\omega t - \frac{\pi}{2}\right)$;
- the voltage on the inductor $V_L = L\frac{dI}{dt} = LI_0 \frac{d}{dt}(\cos(\omega t)) = -L\omega I_0 \sin(\omega t) = R_L I_0 \cos\left(\omega t + \frac{\pi}{2}\right)$.

Vector representation of voltage



The sum of the voltage is equal to external voltage:

$$\vec{V}_R + \vec{V}_C + \vec{V}_L = \vec{V},$$

$$\tan(\phi) = \frac{\omega L - \frac{1}{\omega C}}{R}.$$

Here ϕ is the shift between the current and voltage.
External voltage:

$$U = U_0 \cos(\omega t + \phi),$$

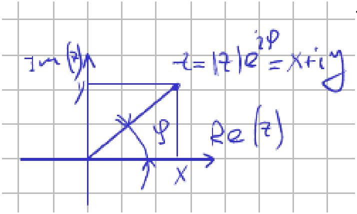
The triangle:

$$(I_0 R)^2 + I_0^2(\omega L - \frac{1}{C\omega})^2 = U_0^2.$$

Then

$$I_0 = \frac{U_0}{\sqrt{R^2 + (\omega L - \frac{1}{C\omega})^2}}.$$

Complex numbers



Algebraic
form of a complex number $z \in \mathbb{C}$

$$z = x + j\,y, \quad x, y, \in \mathbb{R}, \quad j^2 = -1.$$

Complex conjugated:

$$\bar{z} = x - jy.$$

Absolute value

$$|z|^2 = x^2 + y^2, \quad |z|^2 = z\bar{z} = (x + jy)(x - jy) = x^2 + y^2.$$

Argument

$$\mathrm{Arg}(z) = \begin{cases} \arctan\left(\frac{x}{y}\right), & y > 0; \\ \pi + \arctan\left(\frac{x}{y}\right), & y < 0; \\ 0,\ x > 0, y = 0; \\ \pi,\ x < 0, y = 0; \\ \text{undefined},\ x = 0,\ y = 0. \end{cases}$$

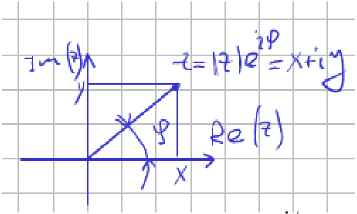
Algebraic form of complex number

$$(x + jy) + (a + jb) = (x + a) + j(y + b),$$

$$(x + jy)(a + jb) = (xa - yb) + j(xb + ya);$$

$$\frac{x + jy}{a + jb} = \frac{(x + jy)(a - jb)}{(a + jb)(a - jb)} = \frac{(xa + yb) + j(ay - xb)}{a^2 + b^2}.$$

Trigonometric form of the complex number



$$z = x + jy = |z|(cos(\phi) + j\sin(\phi)),$$

$$\phi = \mathrm{Arg}(z).$$

Euler's formula:

$$e^{j\phi} = cos(\phi) + j\sin(\phi).$$

$$z = |z|e^{j\phi}.$$

$$z = |z|e^{j\phi}, \quad w = |w|e^{j\psi}, \quad zw = |z||w|e^{j(\phi+\psi)}.$$

$$\frac{z}{w} = \frac{|z|e^{j\phi}}{|w|e^{j\psi}} = \frac{|z|}{|w|}e^{j(\phi-\psi)}.$$

Impedance

Let us fix the law for the voltage: $V = V_0 \cos(\omega t)$. It is convenient to consider complex value

$$v(t) = V_0\, e^{j\omega t}, \quad j^2 = -1.$$

In this case the Ohm law considered as follows

$$ir = v, \quad i = I_0\, e^{j(\omega t - \phi)}.$$

Following parameter is called *impedance* :

$$r = \frac{v}{i} = R_0\, e^{j\phi}, \quad R_0 = \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}$$

The formula

$$v = ir, \quad r \in \mathbb{C}$$

can be considered as the Ohm law in a complex form.

Impedance

The formula for the impedance in the algebraic form:

$$r = R + j\,Z, \quad Z = \omega L - \frac{1}{\omega C}.$$

The phase ϕ can be obtained using a triangle where

$$\tan(\phi) = \frac{Z}{R}.$$

The amplitude of the impedance is given by formula:

$$R_0 = \sqrt{R^2 + \left(\omega L - \frac{1}{C\omega}\right)^2}.$$

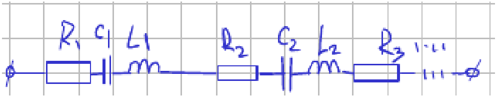
Phasors

In the complex form for both the voltage and the current one can see the multiplier $e^{i\omega t}$:

$$v = V_0\, e^{i\omega t}, \quad i = I_0\, e^{j\,\phi}\, e^{i\omega t}.$$

This multiplier often neglected from the formulas for the current and voltage.
The multiplier $e^{i\omega t}$ is called *phasor*.
Another phasor is the multiplier $e^{j\,\phi}$.

Impedance for serial connection



For serial connected elements of AC circuit the capacity of the circuit:

$$\frac{1}{C\omega} = \sum_l \frac{1}{C_l\omega},$$

the inductivity of the circuit:

$$L = \sum_m L_m\omega,$$

and the resistance:

$$R = \sum_n R_n.$$

Impedance for serial connection

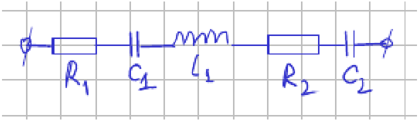
So the impedance is equal

$$r = R + j\left(L\omega - \frac{1}{C\omega}\right).$$

In general case r_k is the impedance of k-th element in the serial connected circuit:

$$r = \sum_k r_k.$$

Example



Resistance:

$$R = R_1 + R_2,$$

reactance is a difference of inductance and capacitance.

Capacitance:

$$\frac{1}{\omega C} = \frac{1}{\omega C_2} + \frac{1}{\omega C_2} = \frac{C_1 C_2}{\omega (C_1 + C_2)},$$

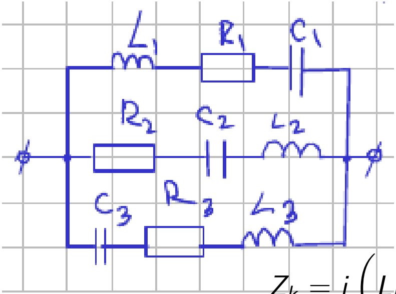
Inductance:

$$L = L_1.$$

Impedance:

$$r = (R_1 + R_2) + j \left(L_1 \omega - \frac{(C_1 + C_2)}{\omega C_1 C_2} \right).$$

Impedance for parallel connection



For parallel connection the same voltage for every line of connection. Every k-th line the complex form of the resistance is defined by

$$r_k = R_k + Z_k,$$

$$Z_k = j\left(L_k\omega - \frac{1}{C_k\omega}\right).$$

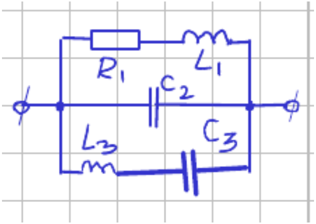
Therefore the Ohm law for every thread:

$$I_k r_k = V, \quad I_k = \frac{V}{r_k}.$$

So for the circuit:

$$I = \sum I_k = \sum \frac{V}{r_k} = V \sum \frac{1}{r_k}.$$

Example 2



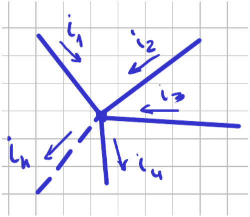
Impedance
of first thread: $r_1 = R_1 + j\omega L_1$.
Impedance
of second thread: $r_2 = -\frac{j}{\omega C_2}$.
Impedance
of third thread: $r_3 = j\left(L_3 - \frac{1}{\omega C_3}\right)$.
Equivalent impedance:

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{r_1 r_2 + r_2 r_3 + r_3 r_1}{r_1 r_2 r_3}.$$

$$r = \frac{(R_1 + j\omega L_1)\frac{j}{\omega C_2} j\left(L_3 - \frac{1}{\omega C_3}\right)}{(R_1 + j\omega L_1)\frac{j}{\omega C_2} + \frac{j}{\omega C_2} j\left(L_3 - \frac{1}{\omega C_3}\right) + (R_1 + j\omega L_1)}.$$

$$r = -\frac{(R_1 + j\omega L_1)(L_1 - \frac{1}{\omega C_3})}{jR_1 - \omega L_1 - L_3 - \frac{1}{\omega C_3} + C_2\omega(R_1 + j\omega L_1)}.$$

Kirchhoff's laws for AC current



For any node of the circuit

$$\sum_k i_k = 0,$$

where i_k is the complex valued AC current on k-th wire.
Every current
can be written in the form:

$$i_k = I_k\, e^{j\phi_k}\, e^{j\omega t}.$$

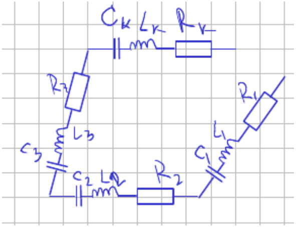
For sinusoidal current this yields:

$$\sum_k I_k\, e^{j\phi_k}\, e^{j\omega t} = 0, \quad e^{j\omega t} \sum_k I_k\, e^{j\phi_k} = 0.$$

In this case one get the sum of the currents for given node as a polygon.

$$\sum_k I_k\, e^{j\phi_k} = 0$$

Kirchhoff's laws for AC current



For every mesh one get:

$$\sum_k \left(i_k R_k + L_k \frac{di_k}{dt} + \frac{1}{C_k} \int i_k dt \right) = \sum_k \mathcal{E}_k,$$

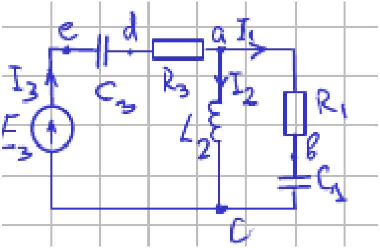
where R_k is resistance,
L_k is a unductance, C_k is a capacitance and i_k is a current through given k-th thread and e_k is a source of voltage.
For the sinusoidal current one get:

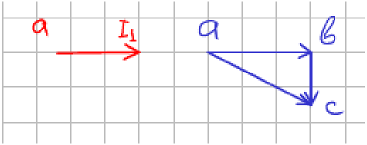
$$\sum_k I_k \left(R_k + L_k \omega e^{j\pi/2} - \frac{1}{\omega C_k} e^{-j\pi/2} \right) e^{j\omega t} = \sum_k e_k \, e^{i\omega t}.$$

$$\sum_k I_k \left(R_k + j \left(L_k \omega - \frac{1}{\omega C_k} \right) \right) = \sum_k e_k.$$

$$\sum_k I_k r_k = \sum_k e_k.$$

Example 3





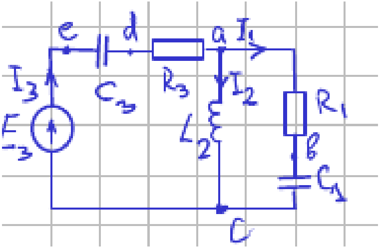
Let us consider the circuit.
Define the positive direction of the current I_1.
The voltage between pins *ab* and *ac* are defined as follows:

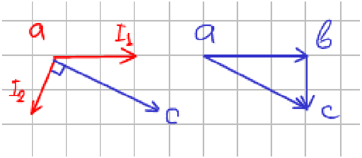
$$V_{ab} = I_1 R_1$$

and

$$V_{ac} = I_1 r_1 = I_1 (R_1 - j\frac{1}{\omega C_1}).$$

Example 3

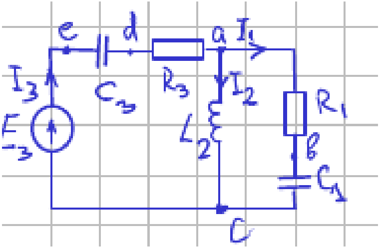


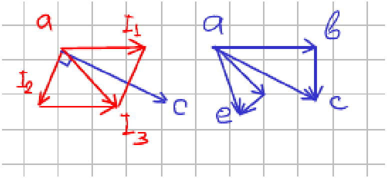


The current I_2 is orthogonal to ac and

$$I_2 = \frac{V_{ac}}{r_{ac}}, \quad r_{ac} = j\omega L_2.$$

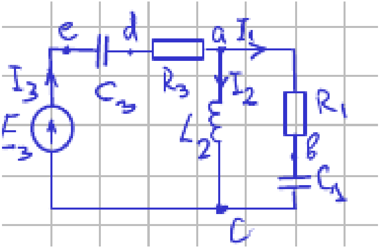
Example 3

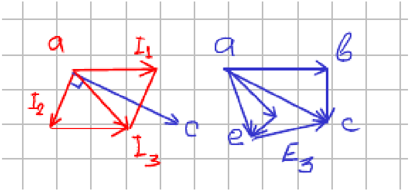




$$I_3 = I_1 + I_2, \quad V_{ad} = I_3 R_3, \quad V_{de} = -I_3 \frac{j}{\omega C_3}, \quad V_{ae} = I_3 \left(R_3 - \frac{j}{\omega C_3}\right).$$

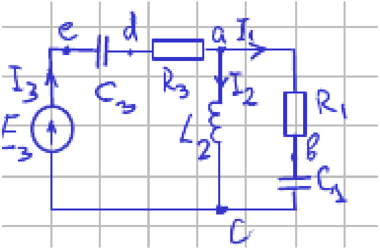
Example 3

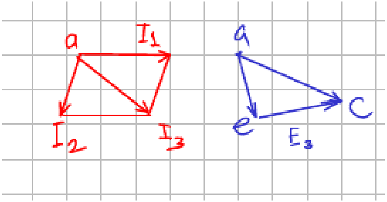




$$E_3 = V_{ac} - V_{ae}.$$

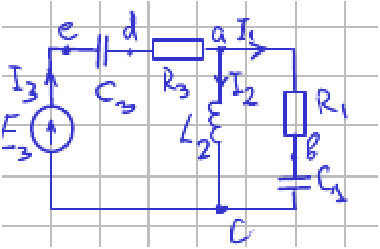
Example 3

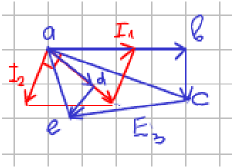




Final graph for the circuit.

Example 3





Two diagrams are united each other.

Summary

- Impedance.
- Kirchhoff laws and diagrams for currents and voltages.