Tutorial 5. Dry friction

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Leonardo da Vinci

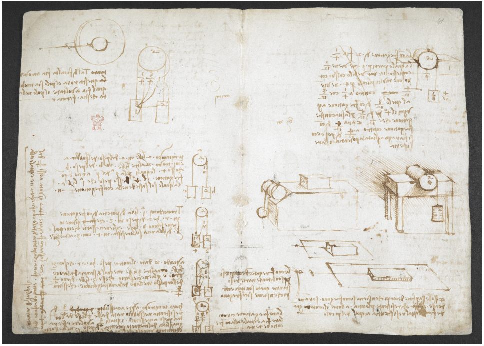


Figure: 1478-1518 Leonardo da Vinci, Notebook "The Codex Arundel". Digitized manuscripts, British Library (Arundel ms 263 f041r.jpg)

Leonardo da Vinci researched the friction between two solid bodies with different stiffness and two bodies with interlayer (1493). Leonardo constructed friction roller bearing which were used in water mills and in lifting mechanisms.

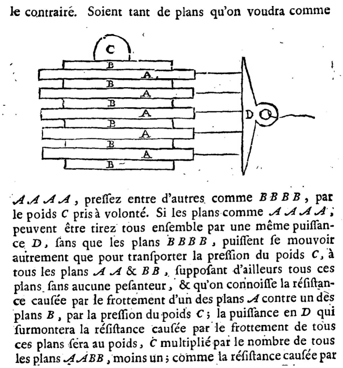
Guillaume Amontons

Guillaume Amontons studied the dry friction between two solid bodies and formulated two laws for dry friction (1699).

- The friction is proportional to the loaded on the surface.
- Friction is independent on the area of contacted surfaces.

Both laws were obtained after study experimental data and both look not trivial. For example about the second law he wrote that there was wrong thinking that the friction grows with the growing of the contact areas.

Amontons's experiment



le contrairé. Soient tant de plans qu'on voudra comme

A A A A, preſſez entre d'autres comme *B B B B*, par le poids *C* pris à volonté. Si les plans comme *A A A A*; peuvent être tirez tous enſemble par une même puiſſance *D*, ſans que les plans *B B B B*, puiſſent ſe mouvoir autrement que pour tranſporter la preſſion du poids *C*, à tous les plans *A A* & *B B*, ſuppoſant d'ailleurs tous ces plans ſans aucune peſanteur, & qu'on connoiſſe la réſiſtance cauſée par le frottement d'un des plans *A* contre un des plans *B*, par la preſſion du poids *C*; la puiſſance en *D* qui ſurmontera la réſiſtance cauſée par le frottement de tous ces plans ſera au poids, *C* multiplié par le nombre de tous les plans *A A B B*, moins un; comme la réſiſtance cauſée par

Figure: M Amonton De La Resistance cause'e dans les machines Histoire de l'Academie royale 1699, 19 Dec.,P. 213

To
proof the independents of the friction on the contact areas he got a pile of several plates and loaded them by some load C. In such construction every plate is under the friction from the upper plate and the lower plate. So the summary force of friction is equal $2F_{fr}$ and the force for movement on n-th plates is equal to

$$F = n2F_{fr}, \quad F_{fr} = Nf \quad F = 2nfN.$$

De La Hire

De La Hire justified the second of the Amontons's friction law by experiment (1700). He also suggested two different the explanations of this phenomenon for elastic and for hard bodies. Let us consider two contacted surfaces as a fields of springs. So the the force to move positions of all springs of the field depends on the number of the springs multiplied by the pressure and the sum of all forces of the field depends only on the load but does not depended on the number of the springs contained the field. If the contact surfaces are hard then due to the roughness of both bodies the load down to the small surface irregularities then for the movement the load move up. But for such vertical movement the weight of the load are important only so the movement does not depend on the contact area.

Euler's contribution

Leonhard Euler at 1748 formulated the dry friction laws and add ones more with respect the laws by Amonton. Namely, Leonhard Euler claim that the force of the dry friction does not dependent on the speed between contacted surfaces. Also Leonhard Euler paid attention on the discontinuous on the speed for the force of friction. In mathematical point of view this force has a first order discontinuity.

Coulomb's law

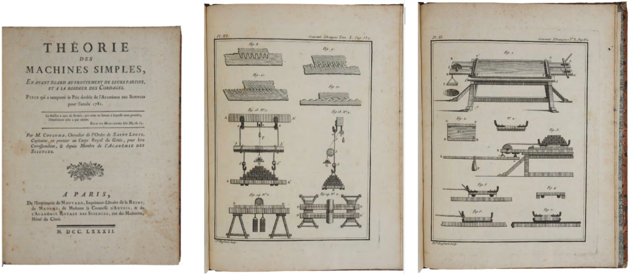


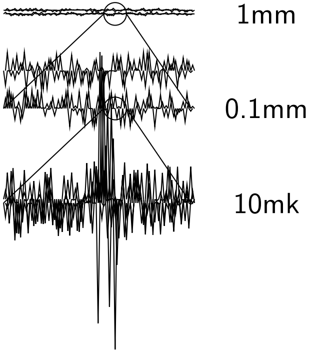
Figure: Colulomb. Theorie des machines simple, Paris,1781

Coulomb made a lot of experiments with the moving bodies under the dry force of friction (1781). His experiments shown the independence the force of friction on the speed of between two bodies. Therefore third law for dry friction was formulated.

- The friction coefficient is independent on the speed between bodies.

Roughness of the surface

Fractal-like surface of contacted bodies



The contact surface can be considered as a fractal with the scaled roughness. The same structure of the roughness repeated on a set of the scales from typical length beginning at 1 mm up to the 100 nm. Review about the roughness and dry friction can be found in J.A.Greenwood and J.J.We 2001

Contact spots

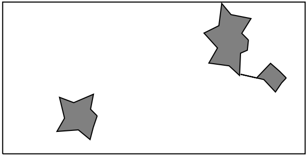
The contact area does not coincide with the visual area of the contact surface of the body. Experiments shows the contact area can be less than 1000 times that the all visible contact area for the body. The force of the friction collect a sum of the friction forces of the contact areas of the contact surfaces.

$$dF_i = f\,\mathrm{sign}(v) p_i dS_i,$$

here p_i – pressure in the i-th contact spot and v is velocity. To obtain value of the friction for the i-th contact spot we must integrate over all spot:

$$F_i = f\,\mathrm{sign}(v) \int_{S_i} p_i dS_i.$$

Contact spots



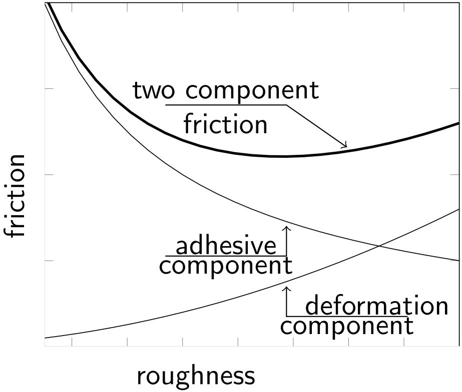
So full force of the friction is a sum over all contact spots:

$$F = \mathrm{sign}(v) f \sum_i \int_{S_i} p_i dS_i, \quad N = \int_{S_i} p_i dS_i.$$

This formula shows that general assumptions are foundation stone for the independence of the friction force with respect to contact area but on the normal load only.
In a lot of experiment shown that the electrical result were approved by Board and Tabor with their experiment (1934)

Two components of the friction force

A scheme friction dendence
on a roughness
(by I.V.Kragelskii
"Friction and wear")



In the contact spot appears adhesive forces on molecular level. The experiments shows that the decreasing of roughness gives decreasing of the friction on the initial stage but next decreasing of the roughness implies to increasing of the friction.

$$f = f_0 + f_a.$$

Here f_0 is the term which defines the deformation forces and f_a connected with all adhesive forces.

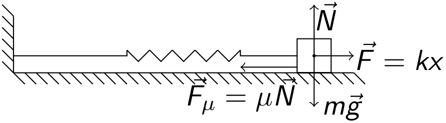
The difference between static and kinematic dry friction

The static dry friction f_s is bigger that kinematic one f_k. This difference is used for example in Anti Blocking System in Cars. If one use emergency braking without blocking the wheels then at any time the contact spot between the wheel and the road are static. Therefore the braking without blocking the wheels are more efficient.

A thought experiment with dry friction

Let us consider a box with the thread on a slanted surface.
For small slant we feel a threshold force to change the position of the box when we pull them on the thread. When the box slides on the slanted surface we can change it direction by any small orthogonal force. This means that movement with such sliding is not stable. The ABS allows to avoid the sliding with blocked wheels and save control on the car when one use emergency braking.

Oscillator with dry friction



If $\dot{x} \neq 0$ then movement of a load with a spring is defined by the following equation:

$$m\ddot{x} = -\mu mg\,\mathrm{sign}(\dot{x}) - kx.$$

If $\dot{x} = 0$, then:

$$m\ddot{x} \in (-\mu mg + kx, \mu mg + kx).$$

Formally it means:

$$m\ddot{x} \in \begin{cases} -\mu mg\,\mathrm{sign}(\dot{x}) - kx, & \dot{x} \neq 0; \\ (-\mu mg + kx, \mu mg + kx), & \dot{x} = 0. \end{cases}$$

Such model is a differential inclusion. The differential inclusion allows us to consider the set $x \in (-\mu mg/k, \mu mg/k)$ and $\dot{x} = 0$ as equilibrium.

Dynamics of the oscillator with dry friction

Consider for simplicity the case $\operatorname{sign}(\dot{x}) \neq 0$. Let us define $\sqrt{k}t/\sqrt{m} = \tau$. It yields:

$$x'' = -x - f\,\mathrm{sign}\left(x'\right), \quad f = \frac{\mu m g}{k}.$$

Then the equation for the movement as $x' > 0$ looks like:

$$x'' = -x - f.$$

Multiply the equation by x' then:

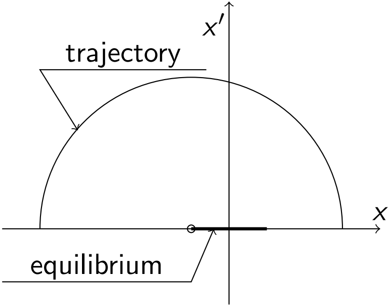
$$x'x'' + x'x = -fx'.$$

This equation we can rewrite as follow:

$$\left(\frac{(x')^2}{2} + \frac{x^2}{2}\right)' = -fx'.$$

This formula shows that the sum in the left-hand side decreases for $x' > 0$.

Dynamics of the oscillator with dry friction



Integration on x yields:

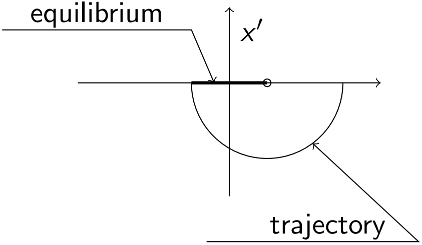
$$\frac{(x')^2}{2} + \frac{x^2}{2} = -fx + E.$$

Here
E is a constant of integration.
We rewrite
previous formula in the form:

$$(x')^2 + (x+f)^2 = f^2 + 2E.$$

That means the trajectory is a semi-circle with the center at $(-f, 0)$ and radius $R = \sqrt{f^2 + 2E}$.

Dynamics of the oscillator with dry friction



In case $\mathrm{sign}(x') < 0$ we obtain:

$$x'' = -x + f.$$

x

After multiplying by x' and integrating we can rewrite as follow:

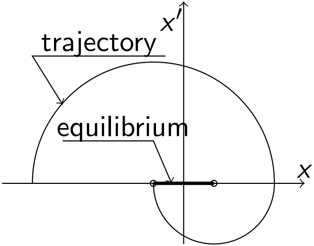
$$\left(\frac{(x')^2}{2} + \frac{x^2}{2}\right)' = fx'.$$

So, the sum in the left-hand side decreases for $x' < 0$ also. After integrating we obtain:

$$(x')^2 + (x - f)^2 = f^2 + 2E_1,$$

That means the trajectory is a semi-circle with the center at $(f, 0)$ and radius $R = \sqrt{f^2 + 2E_1}$.

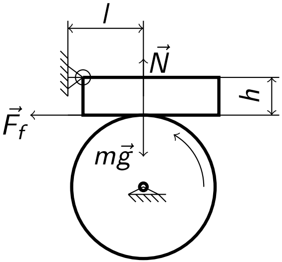
Dynamics of the oscillator with dry friction



Let the initial point of the trajectory be $(x, x') = (x_0, 0)$ where $x_0 < -f$. The part of the trajectory for $x' > 0$ is the semicircle with center at $(-f, 0)$ and radius $r = -f - x_0$. Then the right point of this semicircle $(x_1, 0)$, where $x_1 = x_0 + 2(-f - x_0) = -2f - x_0$

If $x_1 > f$ then this point is initial one for the lower semicircle with left point $(x_1, 0)$ and center at $(f, 0)$ and radius $r = (x_1 - f)$. The left point for this lower semicircle is $(x_2, 0)$, where $x_2 = x_1 - 2(x_1 - f) = -x_1 + 2f = -(-2f - x_0) + 2f = x_0 + 4f$. If $-f \leq x_2$ then the point $(x_2, 0)$ is equilibrium. In opposite case the point $(x_0 + 4f, 0)$ is beginning of the next circle. This next circle begins closer to the equilibrium state then the first circle at $(x_0, 0)$. As the result we get the sequence $\{x_n\}_{k=0}^{n}$ until $x_n \in [-f, f]$.

Painlevé's paradoxes. Example. Brake pad

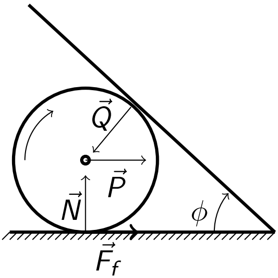


In 1895 Paul Painlevé published critique of the Coulomb's friction law. These thesis are known as Painlevé paradoxes. That critical notes initialized new conceptions of rigid bodies in some problems of dynamics. The simplest of the Painlevé paradox is follows. A friction force: $F_f = \mu N$. A sum of torques: $N\mu h = Nl - mgl$, Therefore: $N(\mu h - l) = -mgl$ As a result we obtain:

$$N = \frac{mgl}{l - h\mu}.$$

This means that for $l = -h\mu$ the reaction of the support is equal infinity.

Painlevé's paradoxes. Disk in a wedge



Let's consider a rotating disk under external horizontal force $\vec{P}$ is situated into an angle with value ϕ. A friction coefficient between the disk and a horizontal plane is equal μ. A friction between upper plane and the disk equals zero. Vertical forces:

$$Q \sin\left(\frac{\pi}{2} - \phi\right) = N,$$

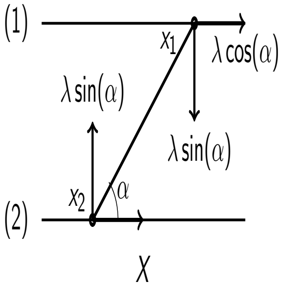
A sum of horizontal forces:

$$P + F_f = Q \cos\left(\frac{\pi}{2} - \phi\right), \quad F_f = \mu N.$$

As a result we obtain:

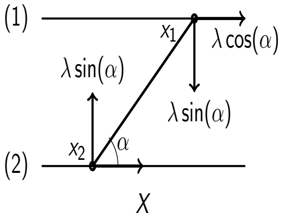
$$P + \mu N = N \tan(\phi) \Rightarrow N = \frac{P}{\tan(\phi) - \mu}.$$

Painlevé's paradoxes. Two beads on rods



Let's consider
two parallel rods (1) and (2) and two beads on both these rods. The beads are connected by a light weight rod.
The low bead slides
along the rod with dry friction μ and the upper bead slides without any friction.
A
horizontal force X acts on the low bead.
Define by λ a force along the connecting rod, $\lambda > 0$ for a contracting force and
$\lambda < 0$ for stretching one.

Painlevé's paradoxes. Two beads on rods



$$
\begin{aligned}
\ddot{x}_1 &= \lambda\cos(\alpha),\\
\ddot{x}_2 &= X - \mu\,\mathrm{sign}(\dot{x}_2)\lambda\sin(\alpha) - \lambda\cos(\alpha),\\
\lambda\cos(\alpha) &= X - \mu\,\mathrm{sign}(\dot{x}_2)\lambda\sin(\alpha) - \lambda\cos(\alpha),\\
\lambda &= \frac{X}{\mu\,\mathrm{sign}(\dot{x}_2)\sin(\alpha) + 2\cos(\alpha)}
\end{aligned}
$$

So, $\lambda = \infty$ when the
angle α and μ are connected by formula;

$$\tan(\alpha) = -\frac{2}{\mu}\,\mathrm{sign}(\dot{x}_2).$$

Discussions

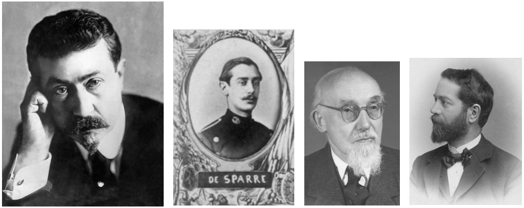


Figure: P. Painlev'e, De Sparre, Hammel, Klein

Figure: Lecornu, Pfeifer,Prandtl, von Mises, Filippov.

Summary

- History of experimental studying the dry friction.
- Physical basics of the dry friction.
- Oscillations under the dry friction.
- Painlevé's paradoxes connected to the dry friction.