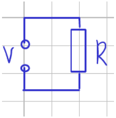
Tutorial 5. Alternating Current.

O.M. Kiselev
o.kiselev@innopolis.ru

Innopolis university

Quasi stable current



Let us consider the current with changing of the direction. The simplest math example of such current can be considered as follows:

$$V(t) = V_0 \cos(\omega t).$$

Here V_0 is an amplitude, and ω is a frequency.
The electric field is changed with the speed of light. That means we can consider the changes as quasi stable if

$$2\pi/\omega \ll l/c.$$

Here l is a typical distance for the circuit and $c = 3 \times 10^8 m/s$ is the speed of light.
So the current trough a resistance R:

$$I(t) = \frac{V(t)}{R_0} = I_R \cos(\omega t), \quad I_0 = \frac{V_0}{R_0}.$$

Power on resistance

Typical loss of the eclectic energy takes place in lines of electric power transmission.
The power on the resistance on current time

$$P(t) = RI^2(t) = RI_0^2 \cos^2(\omega t) = \frac{RI_0^2}{2}(1 + \cos(2\omega t)).$$

The work over time T:

$$W = \int_0^T \frac{RI_0^2}{2}(1 + \cos(2\omega t))dt = \frac{1}{2}RI_0^2 T + \frac{1}{4\omega}RI_0^2 \sin(2\omega T).$$

Power losses

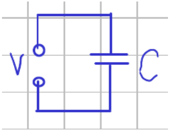
Average power over T:

$$P_a = \frac{1}{T}\int_0^T P(t)dt = \frac{1}{2}RI_0^2 + \frac{1}{4\omega T}RI_0^2\sin(2\omega T),$$

so

$$P_a = \frac{1}{2}RI_0^2, \quad T \to \infty.$$

Capacitor



The capacitor looks
like two plate with dielectric between them.
For
simplicity one consider the linear dependency
of external voltage:The capacity is

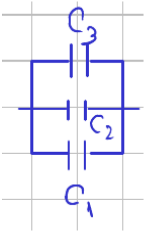
$$Q = CV,$$

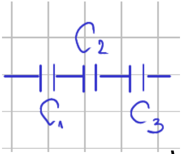
Where C is called *capacitance*. The dimension of the capacitance is *farad*.

$$\left[\frac{C}{V}\right] = [F].$$

The capacitor in the for the direct current disconnected the circuit. In opposite case the AC current charges and discharges the capacitor and as result one can look at this process as AC trough the capacitor.

Connections of the capacitors





The parallel connection

$$Q_i = C_i V$$

$$Q = \sum C_i V = V \sum C_i, \quad C = \sum C_i.$$

The serial connection

$$Q = V_i C_i, \quad V_i = \frac{Q}{C_i},$$

Then the summary voltage:

$$V = \sum \frac{Q}{C_i} = Q \sum \frac{1}{C_i}, \quad \frac{1}{C} = \sum \frac{1}{C_i}.$$

AC through a capacitor

So in this case

$$Q(t) = CV(t),$$

then the current though the capacitor:

$$I_c(t) = \frac{dQ}{dt} = C\frac{dV}{dt}.$$

This formula shows that the AC can pass through the capacitor. The maximum of charge at the time:

$$\frac{dQ}{dt} = 0, \quad I_c(t) = 0.$$

The maximum of the current as

$$\frac{dI_c}{dt} = 0 \quad \text{or} \quad C\frac{d^2V}{dt^2} = 0.$$

AC through a capacitor

Let's say some current through a capacitor $I_c(t)$ the a voltage is:

$$V_c(t) = \frac{1}{C}\int_0^t I_c(t)dt.$$

The power on a capacitor

The formula for the power on the capacitor looks like:

$$P_c = V_c I_c$$

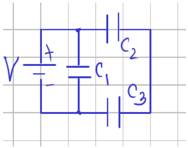
Losses of the power on the capacitor:

$$W = \int_0^T V_c(t) I_c(t) dt = C \int_0^T V_c(t) \frac{dV_c}{dt} dt,$$

then

$$W = C \left(\frac{V_c^2(T)}{2} - \frac{V_c^2(0)}{2} \right).$$

Example 1



Let us consider following circuit. The capacitance of the circuit:

$$C = C_1 + \frac{1}{\frac{1}{C_2} + \frac{1}{C_3}},$$

therefore:

$$C = C_1 + \frac{C_2 C_3}{C_2 + C_3},$$

The value of the energy:

$$W = \frac{1}{2}\left(C_1 + \frac{C_2 C_3}{C_2 + C_3}\right) V^2.$$

Reactive resistance

For the sinusoidal dependency of the voltage:

$$V = V_C \cos(\omega t).$$

One can obtain the current:

$$I(t) = \frac{d}{dt}(CV_C \cos(\omega t)) = -CV_C\,\omega \sin(\omega t) = I_C \cos\left(\omega t + \frac{\pi}{2}\right),$$

where

$$I_C = V_C C\omega = \frac{V_C}{\frac{1}{C\omega}}.$$

Reactive resistance

As well as the following formula

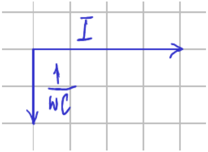
$$I_C = V_C C\omega = \frac{V_C}{\frac{1}{C\omega}}$$

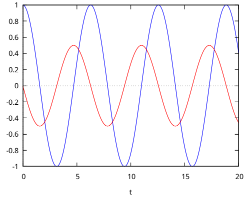
looks like a Ohm law. The value

$$R_C = \frac{1}{\omega C}, \quad [R_C] = \left[\frac{Vs}{C}\right] = \left[\frac{V}{A}\right] = [\Omega] \,.$$

is considered as the reactive resistance of the capacitor.

Phase shift between current and voltage





The voltage:

$$V = V_C \cos(\omega t).$$

The current

$$I(t) = I_C \cos\left(\omega t + \frac{\pi}{2}\right))$$

The phase of the current pass ahead to the phase of voltage on $\pi/2$. Let us consider the direction of the current along the x-axis.

In this case the voltage directed on negative side of y-axis.

Extreme values

The extreme values of the charge at following moments:

$$\frac{dQ}{dt} = 0, \quad \frac{dQ}{dt} = \frac{d}{dt} CV(t) = I_c \sin(\omega t) = 0.$$

It yields:

$$\omega t = \pi n, \quad n \in \mathbb{Z}.$$

The extreme values of the current at following moments:

$$\frac{dI}{dt} = 0, \quad \frac{d^2 Q}{dt^2} = \frac{d^2}{dt^2} CV(t) = -I_c \omega \cos(\omega t) = 0.$$

$$\omega t = \pm \frac{\pi}{2} + 2\pi n, \quad n \in \mathbb{Z}.$$

The work on the capacitor

The work of the sinusoidal current is following:

$$W = \int_0^T V(t)I(t)dt = \int_0^T V_0 \sin(\omega t) V_0 \omega \cos(\omega t)dt =$$

$$= V_0^2 \int_0^T \omega \sin(\omega t)\cos(\omega t)dt = \frac{V_0^2}{2} \int \omega \sin(2\omega t)dt =$$

$$\frac{V_0^2}{4} \int_0^T \sin(2\omega t)d(2\omega t) = \frac{V_0^2}{4}(1 - cos(2\omega T)).$$

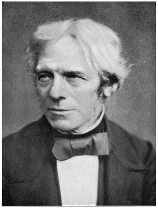
Average power on the capacitor:

$$P_a = \frac{1}{T}\int_0^T P(t)dt = \frac{1}{T}\frac{V_0^2}{4}(1 - cos(2\omega T)).$$

So

$$Pa \to 0, \quad T \to \infty.$$

Inductors



An inductor transform the
electric energy into energy of magnet field.
This phenomenon
was discovered by Faraday (1821).
The movement of a magnet into a bobbin
of conductor generate an electric current
in the circuit which is connected the bobbin.
The dependency of the current and the
voltage:

$$V(t) = L\frac{dI}{dt}.$$

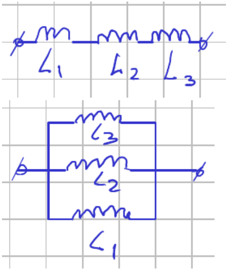
Here L is an inductance.

$$\left[\frac{Vsec}{A}\right] = [H].$$

The value of $I(t)$ with respect to V is following:

$$I(t) = \frac{1}{L}\int V(t)dt.$$

Connection of inductors



A serial connection of inductors:

$$\frac{dI}{dt} = \frac{V_i(t)}{L_i}, \quad \sum V_i = \frac{dI}{dt} \sum L_i,$$

then

$$L = \sum L_i.$$

A parallel connection of inductors:

$$V = \frac{dI_k}{dt} L_k, \quad \frac{dI}{dt} = V \sum \frac{1}{L_k}.$$

Then

$$\frac{1}{L} = \sum \frac{1}{L_k}.$$

Power on the inductor

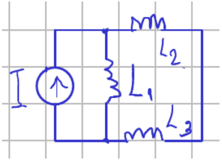
Formula for the power on the inductor:

$$P = VI = LI\frac{dI}{dt}.$$

The losses of the energy on the inductor:

$$W = \int_0^T P(t)dt = L\int_0^T I\frac{dI}{dt}dt = \frac{L}{2}(I^2(T) - I^2(0)).$$

Example 2.



Let us consider following circuit. The capacitance of the circuit:

$$\frac{1}{L} = \frac{1}{L_1} + \frac{1}{L_2 + L_3},$$

therefore:

$$L = \frac{L_1(L_2 + L_3)}{L_1 + L_2 + L_3}.$$

The value of the energy:

$$W = \frac{1}{2}\frac{L_1(L_2 + L_3)}{L_1 + L_2 + L_3} I^2.$$

Sinusoidal current through an inductor

$$I(t) = \frac{1}{L}\int U_L \cos(\omega t)dt = \frac{U_L}{\omega L}\sin(\omega t) + \mathrm{const}\,.$$

The same formula

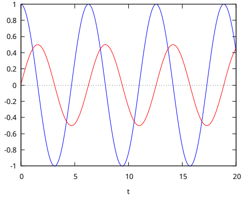
$$I(t) = I_L \cos\left(\omega t - \frac{\pi}{2}\right), \quad I_L = \frac{U_L}{\omega L}.$$

Then one can consider the value

$$R_L = \omega L, \quad [R_L] = \left[\frac{V}{A}\right] = [\Omega]\,.$$

The value ωL is called as *inductive reactance*.

Phase shift between the current and Voltage on the inductor



The voltage on the inductor is:

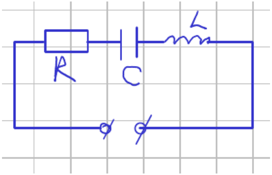
$$V = V_L \cos(\omega t);$$

the current on the inductor is:

$$I = I_L \cos\left(\omega t - \frac{\pi}{2}\right).$$

The phase of the voltage pass ahead to the phase of the current. If direction of the current is x-axis, then the direction of the voltage is positive on the y-axis.

Full resistance

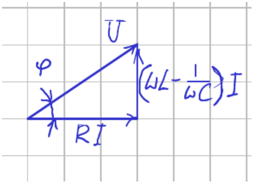


Let us consider serial connection of resistance, capacitor and inductor. In this case for all elements of the circuit the current the same and

$$I = I_0 \cos(\omega t)$$

- The voltage on the resistance: $V_R = I_0 R \cos(\omega t)$. The direction coincides to the current.
- the voltage on the capacitor:
  $V_c = \frac{1}{C} \int_0^t I(t) dt = R_C \cos\left(\omega t - \frac{\pi}{2}\right)$
- the voltage on the inductor $V_L = L \frac{dI}{dt} = R_L \cos\left(\omega t + \frac{\pi}{2}\right)$

Full resistance



The sum of the
voltage is equal to external voltage:

$$\vec{V}_R + \vec{V}_C + \vec{V}_L = \vec{V},$$

$$\tan(\phi) = \frac{\omega L - \frac{1}{\omega C}}{R}.$$

Here ϕ is the shift between the current and voltage.
External voltage:

$$U = U_0 \cos(\omega t + \phi),$$

The triangle:

$$(I_0 R)^2 + I_0^2 (\omega L - \frac{1}{C\omega})^2 = U_0^2.$$

Then

$$I_0 = \frac{U_0}{\sqrt{R^2 + (\omega L - \frac{1}{C\omega})^2}}.$$

Full resistance

The full resistance is given by formula:

$$R_0 = \sqrt{R^2 + \left(\omega L - \frac{1}{C\omega}\right)^2}.$$

If external voltage

$$V = V_0 \cos(\omega t),$$

then the current in this circuit is late:

$$I = I_0 \cos(\omega t - \phi),$$

where

$$I_0 = \frac{U_0}{\sqrt{R^2 + (\omega L - \frac{1}{C\omega})^2}},$$

and

$$\tan(\phi) = \frac{\omega L - \frac{1}{\omega C}}{R}.$$

Summary

- The alternating current through resistance.
- Capacitors and shift between current and voltage.
- Inductors and their properties into circuit.
- Full resistance for circuits with capacitor, inductor and resistance.