Tutorial 4. Newton's law

O.M. Kiselev
o.kiselev@innopolis.ru

Innopolis university

Gravity

Two bodies with masses M_1 and m_2 on the distance R attract to each other by gravitational force as:

$$F = G\frac{M_1 m_2}{R^2}, \quad G = 6.6740810^{-11}\frac{m^3}{kg \times s^2}.$$

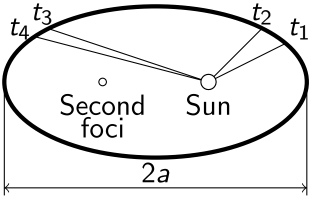
We will try to follow by Isaac Newton step by step to construct and justify one of greatest physical law which was opened on nor on the experiment but on the the tip of the pen by sir Isaac Newton (1686).

Johannes Kepler's laws

A laws of motion for celestial mechanics were knowing before the Newton invention.
Three Kepler's laws on planetary motion were known (1609-1619). There are
1. The orbit planet is an ellipse with the Sun at one of the foci.
2. An area speed of the ellipse segment is a constant.
3. The square of the period of the motion is proportional to the cube of the length of the semi-major axis of the ellipse:

$$T^2 = ka^3, \quad k \in \mathbb{R}.$$

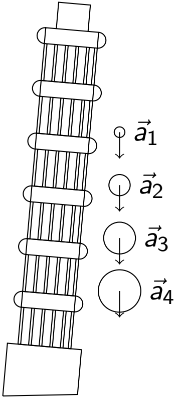
Elliptic trajectory



$$t_2 - t_1 = t_4 - t_3$$
$$S_{t_2,t_1,Sun} = S_{t_4,t_3,Sun}$$

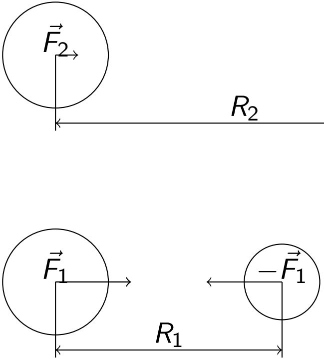
Galileo Galilei's experiments

All of
you know that in the city of Pisa there exists the leaning tower of Pisa. This leaning tower was used by Galileo Galilei (about 1638).
He had dropped balls with the same material but different masses and found that the speed and hence an acceleration of their fall does not depend on their masses.
That means
all things are under action the same force which attracts all bodies to the ground.



$$\vec{a}_1 \equiv \vec{a}_2 \equiv \vec{a}_3 \equiv \vec{a}_4 \equiv \vec{g}.$$

Hook's suppositions



$$R_1 < R_2$$
$$F_1 > F_2$$

Robert Hook formulated the attraction gravitational principles for the free bodies and claimed that there exists a force which attract celestial bodies and the attraction force is stronger for closer bodies (1665). Later he claimed that he had give to Newton the notion about dependency of the gravitation force as reciprocal of squared distances between bodies.

Equations for the motion

The sum of the radial forces

$$m\ddot{r} = mr\dot{\phi}^2 - G\frac{Mm}{r^2}.$$

The equation for the angular velocity:

$$m\frac{d}{dt}(r^2\dot{\phi}) = 0.$$

Then we obtain a formula for the angular velocity

$$mr^2\dot{\phi} = \mu, \quad \mu = \mathrm{const}\,.$$

Here μ is a constant of the integration. This equation allows to write the equation for the radial forces through the non-linear equation of the second order for the function $r(t)$:

$$\ddot{r} = \frac{\mu^2}{m^2 r^3} - G\frac{M}{r^2}.$$

The equations of the motions in the dimensionless form

Let us rewrite the equations of the motion in the simplest form. Define new variables ρ and τ such $r = a\rho$ and $t = b\tau$, the constants a and b are such that the equation for ρ has a simplest form. Substitute new variable into the equation for r:

$$ba\ddot{\rho} = \frac{\mu^2}{m^2 a^3 \rho^3} - G\frac{M}{a^2 r^2}.$$

Then:

$$\frac{\mu^2}{m^2 a^3} = b^2 a, \quad G\frac{M}{a^2} = b^2 a.$$

Or

$$\frac{\mu^2}{m^2} = b^2 a^4, \quad GM = b^2 a^3.$$

$$a = \frac{\mu^2}{m^2 GM}, \quad b^2 = \frac{m^6 G^4 M^4}{\mu^6}$$

The equations of the motions in the dimensionless form

The equation for ϕ has the form:

$$m\frac{\mu^4}{m^4G^2M^2}\rho^2\frac{m^3G^2M^2}{\mu^3}\dot{\phi} = \mu.$$

After all simplification we get two dimensionless equations:

$$\ddot{\rho} = \frac{1}{\rho^3} - \frac{1}{\rho^2},$$

and

$$\rho^2\dot{\phi} = 1.$$

Integration of the equation of motion

Multiply the equation for the radial motion on $\dot{\rho}$:

$$\dot{\rho}\ddot{\rho} = \frac{\dot{\rho}}{\rho^3} - \frac{\dot{\rho}}{\rho^2}.$$

It yields:

$$\frac{d}{d\tau}\left(\frac{\dot{\rho}^2}{2}\right) = -\frac{1}{2}\frac{d}{d\tau}\left(\frac{1}{\rho^2}\right) - \frac{d}{d\tau}\left(\frac{1}{\rho}\right).$$

Or the same form:

$$\frac{d}{d\tau}\left(\frac{\dot{\rho}^2}{2}\right) = \frac{d}{d\tau}\left(-\frac{1}{2}\frac{1}{\rho^2} + \frac{1}{\rho}\right).$$

Integrate the left and right-hand sides of the equation:

$$\frac{\dot{\rho}^2}{2} = -\frac{1}{2}\frac{1}{\rho^2} + \frac{1}{\rho} + E.$$

The first order equations for the motions

Now we obtain two first order equations with additional constant of integration E:

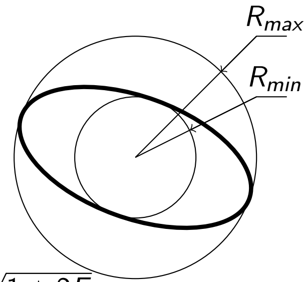
$$\rho^2 \dot{\phi} = 1$$

and

$$\frac{\dot{\rho}^2}{2} + \frac{1}{2}\frac{1}{\rho^2} - \frac{1}{\rho} = E, \quad E = \mathrm{const}\,.$$

The radial motion

It is
easy to understand that the radius changes in the largest and smallest values the speed equals to zero. In that cases we have the following formula fro the values of $\rho = R$:



$$\frac{1}{2}\frac{1}{R^2} - \frac{1}{R} = E$$

or

$$ER^2 + R - \frac{1}{2} = 0, \quad R = \frac{-1 \pm \sqrt{1+2E}}{2E}.$$

For $-1/2 < E < 0$ there exist two different values of R:

$$R_{max} = \frac{-1 - \sqrt{1+2E}}{2E}, \quad R_{min} = \frac{-1 + \sqrt{1+2E}}{2E}.$$

Radial motion

$$\frac{d\rho}{d\tau} = \sqrt{2E + \frac{2}{\rho} - \frac{1}{\rho^2}}.$$

or the equation in the differential form:

$$d\tau = \frac{d\rho}{\sqrt{2E + \frac{2}{\rho} - \frac{1}{\rho^2}}}.$$

Then one can obtain a quadrature:

$$\tau - \tau_0 = \int_{\rho_0}^{\rho} \frac{ds}{\sqrt{2E + \frac{2}{s} - \frac{1}{s^2}}}.$$

The angular motion

The formula for the angle of the body:

$$d\phi = \frac{1}{\rho^2} d\tau$$

or the same using the formula for $d\tau$:

$$d\phi = \frac{\frac{1}{\rho^2} d\rho}{\sqrt{2E + \frac{2}{\rho} - \frac{1}{\rho^2}}}.$$

After integrating we get:

$$\phi - \phi_0 = \int_{\rho_0}^{\rho} \frac{\frac{1}{\rho^2} d\rho}{\sqrt{2E + \frac{2}{\rho} - \frac{1}{\rho^2}}}.$$

Formal integrating

The formal steps to integrate:

$$\int \frac{\frac{1}{\rho^2} d\rho}{\sqrt{2E + \frac{2}{\rho} - \frac{1}{\rho^2}}} = -\int \frac{d\left(\frac{1}{\rho}\right)}{\sqrt{2E + \frac{2}{\rho} - \frac{1}{\rho^2}}} =$$

$$-\int \frac{du}{\sqrt{2E + 2u - u^2}} = -\int \frac{du}{\sqrt{2E + 1 - (1 - 2u + u^2)}} =$$

$$= -\int \frac{du}{\sqrt{2E + 1 - (u-1)^2}} = -\int \frac{d\left(\frac{u-1}{\sqrt{2E+1}}\right)}{\sqrt{1 - \frac{(u-1)^2}{2E+1}}} = \arcsin\left(\frac{u-1}{\sqrt{2E+1}}\right)$$

$$\sin(\phi) = \frac{u-1}{\sqrt{2E+1}}, \quad u = 1 + \sqrt{2E+1}\sin(\phi)$$

$$\rho = \frac{1}{1 + \sqrt{2E+1}\sin(\phi)}.$$

A formula for the trajectory

The formula

$$\rho = \frac{1}{1 + \sqrt{2E+1}\sin(\phi)}.$$

defines the ellipse in polar coordinate with origin at one of foci.

1. The eccentricity $\varepsilon = \sqrt{2E+1}$.
2. Semi-major axe
$$a = \frac{1}{2}\left(\frac{1}{1+\sqrt{2E+1}} + \frac{1}{1-\sqrt{2E+1}}\right) = \frac{1}{-2E}.$$
3. Semi-minor axe can be obtained using the formula $\varepsilon = \sqrt{1 - \left(\frac{b}{a}\right)^2}$. Therefore:
$$4E^2b^2 = -2E, \quad b^2 = -\frac{1}{-2E}, \quad b = \frac{1}{\sqrt{-2E}}.$$
4. Period of rotation equal to the area of the ellipse:
$$T = \int_0^{2\pi} \rho^2 d\phi = \pi ab = \frac{\pi}{\sqrt{(-2E)^3}}, \quad T^2 = \frac{\pi^2}{(-2E)^3}.$$

Earth and Moon

Let us get answer for the question. What are the stronger gravitational force Moon–Sun or Moon–Earth.

- Mass of the Sun $M_s \sim 2 \times 10^{30}$ kg;
- Mass of the Earth $M_e \sim 6 \times 10^{24}$ kg;
- Mass of the Moon $M_m \sim 7.3 \times 10^{22}$ kg;
- The distance between Earth and The Sun $R_{es} \sim 1.5 \times 10^{8}$ km;
- The distance between Moon and Earth $R_{me} \sim 3.8 \times 10^{5}$ km.

Example

The forces:

$$F_{se} = G\frac{M_s M_e}{R_{se}^2} \sim G\frac{2\times 10^{30} 6\times 10^{24}}{1.5^2\times 10^{16}} \sim G\times 5.3\times 10^{38} N,$$

$$F_{sm} = G\frac{M_s M_m}{(R_{se}-R_{me})^2} \sim G\frac{2\times 10^{30} 7.3\times 10^{22}}{1.5^2\times 10^{16}} \sim G\times 6.5\times 10^{36} N,$$

$$F_{em} = G\frac{M_e M_m}{R_{em}^2} \sim G\times\frac{6\times 10^{24}\times 7.3\times 10^{22}}{3.8^2\times 10^{10}} \sim G\times 3\times 10^{36} N.$$