Tutorial 3. Curvilinear motion

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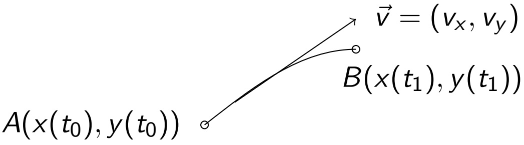
Speed and path

Let us consider the plane motion. Assume that in the Cartesian coordinates the moving point can be written as $x = x(t)$ and $y = y(t)$.



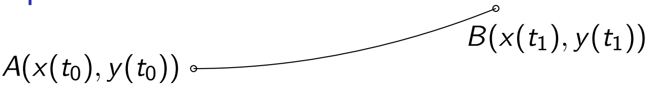
The components of velocity of the this point can be defined as the derivatives with respect to t $v_x = \dot{x}$, $v_y = \dot{y}$.
The speed of the point is the length of the velocity vector:

$$V = \sqrt{v_x^2 + v_y^2}.$$

The length of the path for the trajectory of the point over the period of time $[t_0, t_1]$:

$$S = \int_{t_0}^{t_1} \sqrt{\dot{x}^2 + \dot{y}^2} dt.$$

Examples



Consider the plane motion with constant acceleration:

$$x(t) = v_1 t, \quad y(t) = at^2.$$

The instant velocity at t:

$$v_x = v_1, \quad v_y = 2at.$$

We can thought that the $x(t)$ and $y(t)$ is the parametric form for the trajectory of the point. In that way we obtain, that the vector of instant velocity define the tangent direction at t. The instant speed:

$$V(t) = \sqrt{v_1^2 + 4a^2 t^2}.$$

The length of the path up to the moment t_1:

$$S(t_1) \quad = \int_0^{t_1} \sqrt{v_1^2 + 4a^2 t^2} dt = \frac{t}{2} \sqrt{v_1^2 - 4a^2 t^2} + \frac{v_1^2}{4a} \operatorname{asinh}\left(\frac{2at_1}{v_1}\right).$$

Integrating

$$S(t_1) = \int_0^{t_1} \sqrt{v_1^2 + 4a^2 t^2} dt = v_1 \int_0^{t_1} \sqrt{1 + \frac{4a^2}{v_1^2} t^2} dt =$$

$$\left| \frac{2a}{v_1} t = \sinh(\tau),\ \tau_1 = \mathrm{asinh}(2at_1/v_1),\ dt = \frac{v_1}{2a} \cosh(\tau) d\tau \right| =$$

$$\frac{v_1^2}{2a} \int_0^{\tau_1} \sqrt{1 + \sinh^2(\tau)} \cosh(\tau) d\tau = \frac{v_1^2}{2a} \int_0^{\tau_1} \cosh^2(\tau) d\tau =$$

$$\frac{v_1^2}{8a} \int_0^{\tau_1} (e^{2\tau} + 2 + e^{-2\tau}) d\tau = \frac{v_1^2}{4a} \tau_1 + \frac{v_1^2}{8a} \sinh(2\tau_1)$$

$$\frac{v_1^2}{4a} \tau_1 + \frac{v_1^2}{8a} 2\sqrt{1 - \sinh^2(\tau_1)} \sinh(\tau_1) =$$

$$\frac{v_1^2}{4a} \mathrm{asinh}\left(\frac{2at_1}{v_1}\right) + \frac{v_1^2}{4a} \sqrt{1 - \frac{4a^2}{v_1^2} t^2} \frac{2a}{v_1} t$$

Acceleration and curvature

The acceleration of the point:

$$a_x = \dot{v}_x = \ddot{x}, \quad a_y = \dot{v}_y = \ddot{y}.$$

Theorem

If $\sqrt{v_x^2 + v_y^2} = \mathrm{const}$, *then the acceleration always is orthogonal to the velocity.*

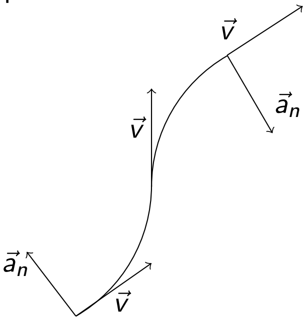
Proof. Let us differentiate the scalar product:

$$\frac{d}{dt}(\vec{v}, \vec{v}) = 0,$$

$$\left(\frac{d}{dt}\vec{v}, \vec{v}\right) + \left(\vec{v}, \frac{d}{dt}\vec{v}\right) = 0$$

$$2\left(\frac{d}{dt}\vec{v}, \vec{v}\right) = 0$$

$$(\vec{a}, \vec{v}) = 0.$$



Acceleration and curvature

Let us consider the circular motion with constant angular velocity:

$$x = R\cos(\omega t), \quad y = R\sin(\omega t).$$

The value of the velocity:

$$v_x = -R\omega\sin(\omega t), \quad v_y = R\omega\cos(\omega t).$$

The formula for the speed looks like:

$$V = \sqrt{R^2\omega^2\sin(\omega t) + R^2\omega^2\cos^(\omega t)} = R\omega.$$

Acceleration and curvature

The acceleration is defined the following formulas:

$$a_x = -R\omega^2 \cos(\omega t), \quad a_y = -R\omega^2 \sin(\omega t).$$

and

$$|a_n| = \sqrt{a_x^2 + a_y^2} = R\omega^2 = \frac{V^2}{R}.$$

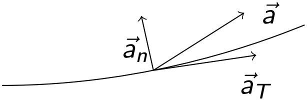
This acceleration is orthogonal with respect to the tangent. The acceleration calls *normal* acceleration. If one know the value of the normal acceleration, then

$$\frac{1}{R} = \frac{|a_n|}{V^2}.$$

The quantity $\rho = 1/R$ is called a curvature.

Tangent acceleration

If the speed changes, then the acceleration might be represented as two orthogonal vectors as the tangent to the trajectory and the normal vector to the trajectory.



The value of the tangent acceleration can be obtained as follows:

$$|a_T| = \frac{(\vec{a}, \vec{v})}{\sqrt{(\vec{v}, \vec{v})}}.$$

The vector of the tangent acceleration:

$$\vec{a}_T = \frac{(\vec{a}, \vec{v})}{(\vec{v}, \vec{v})}\vec{v}.$$

The same formula in the coordinate form looks like:

$$\vec{a}_T = \frac{a_x v_x + a_y v_y}{v_x^2 + v_y^2}(v_x \vec{i} + v_y \vec{j});$$

Normal acceleration

The vector of normal acceleration:

$$\vec{a}_n = \vec{a} - \vec{a}_T.$$

The same formula in the coordinate form is follows:

$$\vec{a}_n = \frac{1}{v_x^2 + v_y^2}(a_x(v_x^2 + v_y^2)\vec{i} + a_y(v_x^2 + v_y^2)\vec{j} - (a_x v_x + a_y v_y)v_x\vec{i} - (a_x v_x + a_y v_y)v_y\vec{j}) = \frac{(a_x v_y - a_y v_x)}{v_x^2 + v_y^2}(v_y\vec{i} - v_x\vec{j})$$

The value of the normal acceleration is:

$$|a_n| = \sqrt{(a, a) - (a_T, a_T)} = \frac{|a_x v_y - a_y v_x|}{|\vec{v}|}.$$

The formula for curvature of the trajectory:

$$\rho = \frac{|a_n|}{(\vec{v}, \vec{v})} = \frac{\sqrt{(a, a) - (a_T, a_T)}}{(\vec{v}, \vec{v})} = \frac{|a_x v_y - a_y v_x|}{(\vec{v}, \vec{v})^{3/2}} = \frac{|\vec{a} \times \vec{v}|}{(\vec{v}, \vec{v})^{3/2}}.$$

Acceleration and curvature. Example

Consider the flat motion with constant acceleration:

$$x(t) = v_1 t, \quad y(t) = at^2.$$

The instant velocity at t:

$$v_x = v_1, \quad v_y = 2at.$$

The instant acceleration:

$$a_x = 0, a_y = 2a.$$

The value of tangent acceleration is

$$|a_T| = \frac{(\vec{a}, \vec{v})}{\sqrt{(\vec{v}, \vec{v})}} = \frac{4a^2 t}{\sqrt{v1^2 + 4a^2 t^2}}.$$

The normal acceleration:

$$|a_n| = \sqrt{4a^2 - \frac{16a^4 t^2}{v1^2 + 4a^2 t^2}} = \frac{2av_1}{\sqrt{v1^2 + 4a^2 t^2}}.$$

Acceleration and curvature. Example

The curvature of the trajectory:

$$\rho = \frac{2av_1}{(v_1^2 + 4a^2t^2)^{3/2}}.$$

Therefore the maximum of the curvature is:

$$\rho_{max} = \frac{2a}{v_1^2},$$

and the curvature tends to zero as $t \to \infty$. The tangent acceleration is $|a_T| = \frac{4a^2t}{\sqrt{v1^2+4a^2t^2}} \to 2a$ as $t \to \infty$. The normal acceleration $\frac{2av_1}{\sqrt{v1^2+4a^2t^2}} \to 0$ as $t \to \infty$.

General formulas

The radius-vector for the trajectory is

$$\vec{r} = (x(t), y(x), z(t)).$$

The velocity for the moving point is tangent to the curve and:

$$\vec{v} = \frac{d}{dt}\vec{r} = (\dot{x}, \dot{y}, \dot{z}).$$

The acceleration is:

$$\vec{a} = \frac{d^2}{dt^2}\vec{r} = (\ddot{x}, \ddot{y}, \ddot{z}).$$

Acceleration on three dimension

The tangent acceleration:

$$\vec{a}_T = \frac{(\vec{a}, \vec{v})}{(\vec{v}, \vec{v})}\vec{v} = \frac{a_x v_x + a_y v_y + a_z v_z}{v_x^2 + v_y^2 + v_z^2}(v_x\vec{i} + v_y\vec{j} + v_z\vec{k}).$$

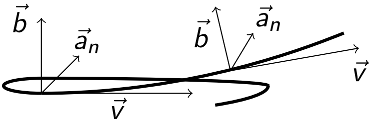
The normal acceleration:

$$\vec{a}_n = \vec{a} - \vec{a}_T.$$

The normal acceleration and tangent vectors define the *osculating plane*. Define a unit vectors $\vec{u} = \frac{\vec{v}}{\sqrt{(\vec{v},\vec{v})}}$ and $\vec{n} = \frac{\vec{a}_n}{\sqrt{(\vec{a}_n,\vec{a}_n)}}$. The vector $\vec{b} = \vec{u} \times \vec{n}$ is called *binormal*. The vectors $\vec{u}, \vec{n}, \vec{b}$ define the orthogonal system of the vectors connected with the curve.

The torsion of the trajectory

Torsion is a derivative of the angle of rotation of osculating plane with respect to changing the length of the curve.



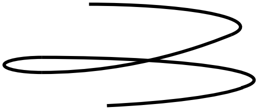
The normal vector to the osculation plane:

$$\vec{b} = \vec{v} \times \vec{a}.$$

The formula for the torsion has the form:

$$\tau = |\dot{\vec{b}}| \frac{dt}{dl}.$$

The torsion and trajectory. An example helix.

Helix: 

$$\vec{r} = (cos(t), \sin(t), t), \quad \vec{v} = (-\sin(t), \quad , \cos(t), 1)$$

$$\vec{a} = (-\cos(t), -\sin(t), 0); \quad \vec{b} = \vec{v} \times \vec{a} = (-\sin(t), -\cos(t), 0).$$

$$\dot{\vec{b}} = (-\cos(t), \sin(t), 0), \quad dl = \sqrt{\sin^2(t) + \cos^2(t) + 1} dt.$$

Torsion:

$$\tau = |\dot{\vec{b}}| \frac{dt}{dl} = \frac{\sqrt{\cos^2(t) + \sin^2(t)}}{\sqrt{\sin^2(t) + \cos^2(t) + 1}} = \frac{1}{2}.$$

Summary

- A length of a path is defined curvilinear integral of the second kind.
- Value of normal acceleration and speed define the curvature of a curve.
- Binormal vector define the osculating plane.
- Torsion define the change of the binormal vector along the trajectory.