Tutorial 2. Kirchhoff's laws and graph theory.

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Kirchhoff's laws for electric circuit
- History
- Ohm's law
- Kirchhoff's current law
- Kirchhoff's voltage law
- Summary

Georg Ohm

Figure: Georg Ohm 1789-1854

Georg Ohm was a scholar teacher.

- His famous work about resistivity was published in his book at 1827, but it was recognized only at 1841 year.
- An acoustic law about expansion of a sound on single phases also has his name.

Gustav Kirchhoff

Figure: Gustav Kirchhoff 1824-1887

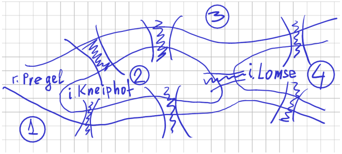
The name of G. Kirchhoff is connected with several physical phenomena. The list of these laws contains the following:

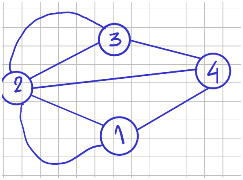
- Kirchhoff's circuit laws.
- Kirchhoff's law of thermal radiation
- Kirchhoff's law of thermochmistry
- Kirchhoff's laws of spectroscopy.

Leonhard Euler

Figure: Leonhard Euler 1707-1783

Leonhard Euler's famous problem about bridges was an origin of the graph theory.





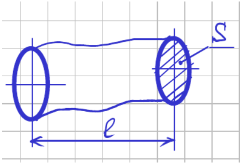
Resistance and resisitivity

The word **resistivity** means the electrical properties of a material as a conductor with length 1 meter and cross section equal to 1 meter^2. Often the resistivity is written as ρ. The physical dimension of this property is $\rho = [\Omega \times m]$.
That means the word resistivity is a property of a given conductor.
The resistivity depends on a temperature. Typically the value of the resistivity is given at $293K \sim 20^{\circ}$C.
Therefore to find the resistivity at a certain temperature one must multiply the resistivity with the temperature coefficient $\kappa[K^{-1}]$.
The resistivity with temperature correction can be written as follows:

$$\tilde{\rho} = \rho\kappa(T - 293).$$

The values of the ρ and κ are tabulated and one can find these values in physical handbook for different materials.

Resistance and resisitivity



Another
word which is used is the word **resistance**. This word means the value of an electrical resistor R the same resistance may be obtained using a different kind of material. It depends on the length l and area of the cross section S and as temperature $T[K]$:

$$R = \rho \frac{l}{S} \kappa (T - 293)[\Omega].$$

One often neglects the temperature coefficient as a result the formula looks simpler:

$$R = \rho \frac{l}{S} [\Omega].$$

Typical example for resistivity

One had measured a resistance $R \sim 0.032\Omega$ of the wire length $2m$ and cross section $S = 1mm^2 = 10^{-6}m^2$. What is the resistivity of the wire?

$$\rho = R\frac{S}{l} = 0.032\frac{10^{-6}}{2} = 1.6 \times 10^{-8}[\Omega \times m].$$

Using a handbook one can suppose that the material of the wire is the silver (Ag).

Example for the resistivity

Les us consider the simplest question the resistivity of the Al wire is $\rho_{al} = 2.65 \times 10^{-8} [\Omega \times m]$ and the resistivity of the Cu is $\rho_{Cu} = 1.68 \times 10^{-8} [\Omega \times m]$.
What is the relation between the cross sections for the Al and Cu to be equal resistance if the length of the wires is the same?
The resistance of the Al and Cu wires of the length l_{Al}, l_{Cu} and cross sections S_{Al}, S_{Cu} are:

$$R_{Al} = \rho_{Al} \frac{l_{Al}}{S_{Al}}, \quad R_{Cu} = \rho_{Cu} \frac{l_{Cu}}{S_{Cu}}.$$

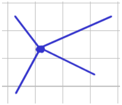
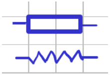
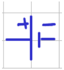
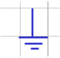
Therefore

$$\rho_{Al} \frac{l_{Al}}{S_{Al}} = \rho_{al} \frac{l_{Cu}}{S_{Cu}}.$$

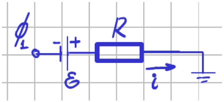
Then the fraction S_{Al}/S_{Su} is following:

$$\frac{S_{Al}}{S_{Su}} = \frac{\rho_{al}}{\rho_{Cu}} = \frac{2.65 \times 10^{-8}}{1.68 \times 10^{-8}} \sim 1.58.$$

Definitions

-  Node.
-  Resistor.
-  Source of electromotive force.
-  Ground.

The Ohm law

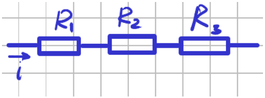


Let us consider one thread of a circuit. This thread contains a pin with given value of the voltage ϕ_1 an resistor with given value R_1 an additional electromotive force $\mathcal{E}_1$ and the last element as a ground. The voltage of the ground is considered as zero.
In this case the formula for changing of the voltage looks like:

$$\phi_1 + \mathcal{E}_1 - i_1 R_1 = 0.$$

If one know three other values then one will be able to obtain the other one.

Serial connection of resistors



The fall of the voltage
on the resistor with resistance R is equal to

$$\Delta u = Ri.$$

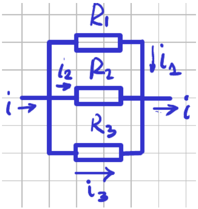
If one has a circuit with serially connected resistors then the falling voltage can be calculated as follows

$$U = \sum \Delta u_k = \sum iR_k = i \sum R_k = iR,$$

Here

$$R = \sum R_k.$$

Parallel connected resistors



For parallel connected resistors the voltage are the same in all parallel threads. Let us define the resistance of the cut of the circuit as R, then

$$u = iR, \quad i = \frac{u}{R}, \quad i = \sum i_k.$$

For all parallel thread

$$u = R_k i_k, \quad i_k = \frac{u}{R_k}.$$

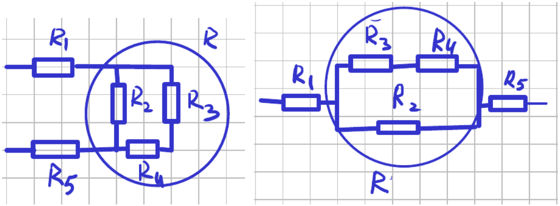
So one has the following formula:

$$\frac{u}{R} = \sum \frac{u}{R_k} = u \sum \frac{1}{R_k}.$$

It yields the final formula resistance of the parallel connection:

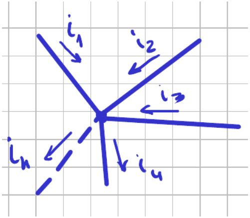
$$\frac{1}{R} = \sum \frac{1}{R_k}.$$

General rule for resistance



Finding a resistance of circuit with complex topology one should dissect the circuit on a set of circuits with parallel and sequence connected of the resistors.

Kirchhoff's current law (KCL)

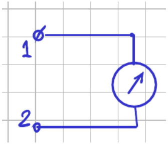


For any node (pin) a
sum of all oriented current is equal to zero.

$$\sum_{k=1}^{n} i_k = 0$$

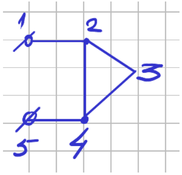
Value of the current which is tended to the node is positive and otherwise is negative.

The application of the KCL



If one can register the
currents only, how many devices one must have to define the currents for all threads? For the simplest
case we obviously need only one device.

The application of the KCL-2



For more complex case like in the figure one can decide using the sequence of the equations.

$$\begin{aligned} i_{12} - i_{23} - i_{24} &= 0, \\ i_{23} - i_{34} &= 0, \\ i_{24} + i_{34} - i_{45} &= 0. \end{aligned}$$

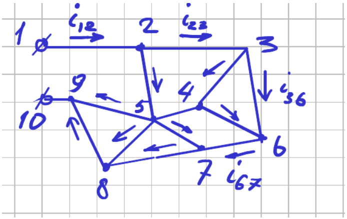
This system can be represented in a matrix form:

$$Ai = 0, \quad i = [i_{12}, i_{23}, i_{24}, i_{34}, i_{45}],$$

$$A = \begin{pmatrix} 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix}, \quad \mathrm{rank}(A) = 3.$$

This means one needs only two measurements (where?) for defining of current in every thread of the circuit.

The application of the KCL-3



At last let us
consider one more complex example.

$$
\begin{aligned}
i_{12} - i_{23} - i_{25} &= 0,\\
i_{23} - i_{34} - i_{36} &= 0,\\
i_{34} - i_{46} - i_{45} &= 0,\\
i_{25} + i_{45} - i_{57} - i_{58} - i_{59} &= 0,\\
i_{36} + i_{46} - i_{67} &= 0,\\
i_{57} + i_{67} - i_{78} &= 0,\\
i_{58} + i_{78} - i_{89} &= 0,\\
i_{59} + i_{89} - i_{910} &= 0.
\end{aligned}
$$

The matrix connected with this system of equations has a dimension 8×14 and the rank of this matrix is equal 8. Therefore one should use only 14-8=6 measurement for defining values of the currents at all threads.

KLC, graphs and nodes

In general case the electric circuit should be considered as a graph. The pins are the nodes of the graph and conductors are the edges of the graph.

- Building the system of equations uses all nodes of the graph.
- Every edge connected with given node is an origin of a current.
- The direction of the current is indicated as an oriented edge.
- Every node of the graph defines an equation contained the values of the current for edges connected with given node.
- The current in the given conductor appears in two equations only. The sings of the same current in these equations are opposite.
- A difference between numbers of the edges and a rank of the matrix gives a necessity quantity of measurements for a defining value of the current in every conductor of the circuit.

Kirchhoff's voltage law (KVL)

Let us consider the simplest circuit as a loop containing resistors and a source of the voltage. In this case the right-hand side has the same value of the voltage: ϕ_1.

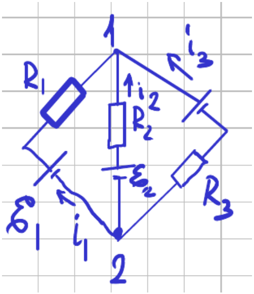
The measuring of voltage one must use a voltmeter in parallel case. In this case changing of the voltage can be found as following:

$$\phi_1 + \mathcal{E} - iR = \phi_1, \quad \mathcal{E} - iR = 0.$$

Here the $\mathcal{E}$ is an electromotive force. So if one know two quantities form three given in the equation one will be able to define another one:

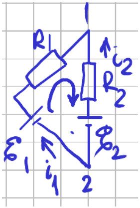
$$\mathcal{E} = iR, \quad R = \frac{\mathcal{E}}{i}, \quad i = \frac{\mathcal{E}}{R}.$$

KVL in case of more than one loop



The complex case for the KVL contains several loops in the circuit. Next case for the circuit we consider the follow figure. Here one can find three different loops.

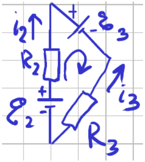
KVL in case of more than one loop



First loop contains the elements R_1, $\mathcal{E}_1$ and R_2, $\mathcal{E}_2$. The formula for the voltage of this loop can be written as follows:

$$\mathcal{E}_1 + i_1 R_1 - \mathcal{E}_2 - i_2 R_2 = 0.$$

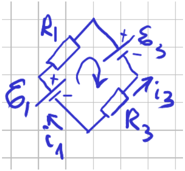
KVL in case of more than one loop



The similar equation
can be written for the following loop:

$$\mathcal{E}_2 + i_2 R_2 - \mathcal{E}_3 - i_3 R_3 = 0.$$

KVL in case of more than one loop



The last loop can be written in the form:

$$\mathcal{E}_1 + i_1 R_1 - \mathcal{E}_3 - i_3 R_3 = 0.$$

KLV in case of more than one loop

Here one obtains three equations for one circuit. These equations connect the value of current, electromotive force and resistance.

$$\mathcal{E}_1 + i_1 R_1 - \mathcal{E}_2 - i_2 R_2 = 0,$$
$$\mathcal{E}_2 + i_2 R_2 - \mathcal{E}_3 - i_3 R_3 = 0,$$
$$\mathcal{E}_1 + i_1 R_1 - \mathcal{E}_3 - i_3 R_3 = 0.$$

Only two of these equations are linear independent. If one subtracts the third equation from the first equation, then one will get the second equation.

KCL in case of more than one loop

The equations for the current in the node 1 and node 2 are:

$$i_1 + i_2 + i_3 = 0,$$
$$-i_1 - i_2 - i_3 = 0.$$

Here only one of these equations is independent.
This means the full system of the equations for the circuit has a $rank = 3$. In this equation one has 9 parameters R_k, i_k, u_k, $k \in \{1, 2, 3\}$. Therefore generally speaking one can consider only 3 of them as unknown.

Example 1

$$\begin{aligned} \mathcal{E}_1 + i_1 R_1 - \mathcal{E}_2 - i_2 R_2 &= 0, \\ \mathcal{E}_2 + i_2 R_2 - \mathcal{E}_3 - i_3 R_3 &= 0, \\ i_1 + i_2 + i_3 &= 0. \end{aligned}$$

Let us consider the case for the following given values

$$\mathcal{E}_1 = 0, \quad R_1 = 2\Omega, \quad \mathcal{E}_2 = 4V, \quad R_2 = 4\Omega, \quad \mathcal{E}_3 = 0, \quad R_3 = 1\Omega.$$

In this case one get:

$$\begin{pmatrix} 2 & -4 & 0 \\ 0 & 4 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ 0 \end{pmatrix}.$$

$$(i_1, i_2, i_3) = (2/7, -6/7, 4/7).$$

Example 2

$$\mathcal{E}_1 + i_1 R_1 - \mathcal{E}_2 - i_2 R_2 = 0,$$
$$\mathcal{E}_2 + i_2 R_2 - \mathcal{E}_3 - i_3 R_3 = 0,$$
$$i_1 + i_2 + i_3 = 0.$$

Let us consider the case for the following given values

$$i_1 = 2A, \quad R_1 = 2\Omega, \quad i_2 = 1A, \quad R_2 = 4\Omega, \quad \mathcal{E}_3 = 0V, \quad R_3 = 1\Omega.$$

In this case one get:

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 \\ i_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -4 \\ -3 \end{pmatrix}.$$

$$(\mathcal{E}_1, \mathcal{E}_2, i_3) = (-7, -7, -3).$$

Example 3. The answer is physically nonsense

$$\begin{aligned} \mathcal{E}_1 + i_1 R_1 - \mathcal{E}_2 - i_2 R_2 &= 0, \\ \mathcal{E}_2 + i_2 R_2 - \mathcal{E}_3 - i_3 R_3 &= 0, \\ i_1 + i_2 + i_3 &= 0. \end{aligned}$$

Let us consider the case for the following given values

$$i_1 = 2A, \quad R_1 = 2\Omega, \quad \mathcal{E}_2 = 0V, \quad R_2 = 4\Omega, \quad \mathcal{E}_3 = 1V, \quad i_3 = 1A.$$

In this case one get:

$$\begin{pmatrix} 1 & -4 & 0 \\ 0 & 4 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{E}_1 \\ i_2 \\ R_3 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \\ -3 \end{pmatrix}.$$

$$(\mathcal{E}_1, i_2, R_3) = (-16, -3, -13).$$

Example 4. Undefined case

$$\mathcal{E}_1 + i_1 R_1 - \mathcal{E}_2 - i_2 R_2 = 0,$$
$$\mathcal{E}_2 + i_2 R_2 - \mathcal{E}_3 - i_3 R_3 = 0,$$
$$i_1 + i_2 + i_3 = 0.$$

Let us consider the case for the following given values

$$i_1 = 2A, \quad R_1 = 2\Omega, \quad i_2 = -1A, \quad R_2 = 4\Omega, \quad R_3 = 1\Omega, \quad i_3 = -1A.$$

In this case one get:

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 \\ \mathcal{E}_3 \end{pmatrix} = \begin{pmatrix} -8 \\ 3 \end{pmatrix}.$$

$$(\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3) = (\mathcal{E}_3 - 5, 3 + \mathcal{E}_3, \mathcal{E}_3).$$

General case

Let us consider the electric circuit as a graph with given given nodes k, oriented edges with flows on the edges as values of current i_k, weights as resistances R_k and electromotive force $\mathcal{E}_k$. The calculations for the circuit look like solution of a system of equations which are

- the equation of Kirchhoff's current law for every node;
- the equation of Kirchhoff's voltage law for every mesh.
- For given values of parameters of circuit one can obtain the unknown quantities for given circuit by analysis of the system of equations.

Problem

Let us define a set of edges of the graph like

$$\{n_k \& EXX \& RXX \& iXX \& n_{k+1}\}_{k=1}^{N}.$$

Here

- n_k is an initial node of the edge,
- EXX is a value of the electromotive force,
- RXX is a value of the resistance,
- iXX is a value of the current,
- n_{k+1} is the final node of this edge.

Instead of the characters XX may be digits or sign ? if the value is unknown.

Calculate the unknown quantities if it is possible.

Summary

- Resistivity and resistance.
- Ohm's law and rules for calculation of the resistance.
- Kirchhoff's law for current in a node.
- Kirchhoff's law for a loop.
- Connection of the Kirchhoff's laws and graph theory.
- Further examples