# Tutorial 1. Linear motion

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Motion on the straight line
- Coordinate, velocity, acceleration
- Differential equation for moving
- Motion with acceleration
- Phase plane

# Straight line motion

Let us consider the dependency of point on the coordinate axis as $x(t)$. Then we can construct the graph for this dependency on the plane $(t, x)$.

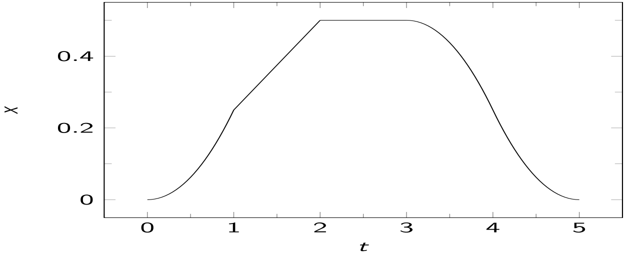
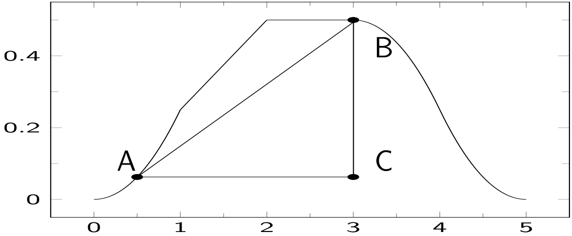



Figure: The example of a dependency of the coordinate $x$ on $t$. On initial interval $x \in (0, 1)$ and $x(t) = (t/2)^2$, on second interval $x \in (1, 2)$ and $x = t/4$, on third interval $t \in (2, 3)$ and $x(t) = 1/2$, on forth interval $t \in (3, 4)$ and $x = \frac{1}{2} - \frac{1}{4}(t - 3)^2$, on the final interval $t \in (4, 5)$ and $x(t) = \frac{1}{4}(t - 5)^2$.

# The average velocity

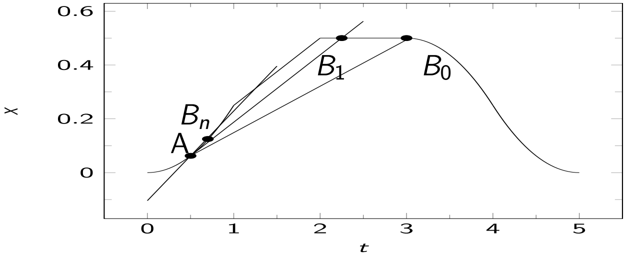



The average velocity for the $x(t)$ on the time interval between $t_A$ and $t_B$ defines as the fraction:

$$\overline{v}_{1,2} = \frac{x(t_B) - x(t_A)}{t_B - t_A} = \frac{|BC|}{|AC|}.$$

The average velocity is an angle coefficient for the straight line $AB$.

# Instant velocity




Let us consider the sequence of moments $t_k$ such that
$t_A < \cdots < t_{k+1} < t_k < \cdots < t_B$, $t_k \to t_A$ as $k \to \infty$.
Thus we obtain the sequence $B_k = (t_{B_k}, x(t_{B_k}))$ and $B_k \to A$, as $k \to \infty$.

# Instant velocity

So we obtain the following right hand-side limit:

$$v_+(t_A) = \lim_{t \to t_A + 0} \frac{x(t) - x(t_A)}{t - t_A}.$$

The same sequence we can use for the left hand-side limit

$$v_-(t_A) = \lim_{t \to t_A - 0} \frac{x(t) - x(t_A)}{t - t_A}.$$

If at the moment $t_A$ these limits are equal: $v_- = v_+$, then the limit calls as **instant velocity**:

$$v(t) \equiv \dot{x}(t).$$

The upper dot defines the derivative with respect to $t$. In mechanics and physics this notation origins from I. Newton.

# Instant velocity

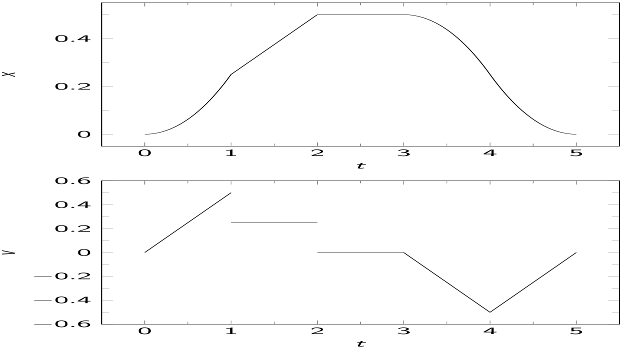



Figure: The instant velocity $v \equiv \dot{x}(t)$ is obtained by differentiating of the graph for $x(t)$. At the discontinuity points the velocity does not defined. On initial interval $t \in (0, 1)$ and $v(t) = (t/2)$, on second interval $t \in (1, 2)$ and $v = 1/4$, on third interval $t \in (2, 3)$ and $v(t) \equiv 0$, on forth interval $t \in (3, 4)$ and $v(t) = -\frac{1}{2}(t - 3)$, on the final interval $t \in (4, 5)$ and $v(t) = \frac{1}{2}(t - 5)$.

# The acceleration

- The acceleration of the point is $a(t) \equiv \dot{v}(t)$ or $a(t) = \ddot{x}(t)$.

To obtain the acceleration one should make the same steps with respect to the velocity as these steps for the velocity with respect to the coordinate.

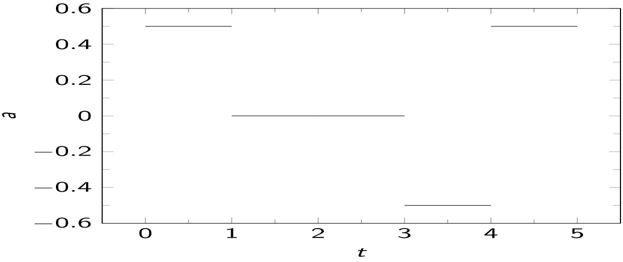



Figure: The instant acceleration $a(t) = \dot{v} \equiv \ddot{x}(t)$ is obtained by differentiating of the graph for $v(t)$. At the discontinuity points the acceleration does not defined. On initial interval $t \in (0, 1)$ and $a = 1/2$, on second interval $t \in (1, 2)$ and $a(t) \equiv 0$, on third interval $t \in (2, 3)$ and $a(t) \equiv 0$, on forth interval $t \in (3, 4)$ and $a(t) = -\frac{1}{2}$, on the final interval $t \in (4, 5)$ and $a(t) = \frac{1}{2}$.

# The differential equation for moving

Let us consider the inverse case when we know the velocity $v(t) \equiv f(t)$ of the point $x$.
That means, we know the differential equation of the first order for the motion:

$$\dot{x} = f(t).$$

If we know a position $x_0$ of the point at $t = t_0$, then we can obtain the trajectory:

$$x(t) = x_0 + \int_{t_0}^{t} f(\tau) d\tau.$$

# Motion with known acceleration

The Newton's laws define the motion of the material point. These laws contains the acceleration and the position of the material point. Therefore generally we have the equation of the motion like follow:

$$\ddot{x} = a(t),$$

where $a(t)$ is an acceleration. This equation can be rewritten in form of two differential equations:

$$\dot{x} = v(t), \quad \dot{v} = a(t),$$

here $v(t)$ is the velocity.
If we know the position $x_0$ and velocity $v_0$ at some $t = t_0$ then we can obtain the position $x(t)$:

$$v(t) = v_0 + \int_{t_0}^{t} a(\tau) d\tau, \quad x(t) = x_0 + \int_{t_0}^{t} v(\tau) d\tau.$$

# Motion with constant acceleration

Let us suppose

$$a = a_0 \in \mathbb{R}.$$

Let the initial values of the velocity and coordinate be

$$v\big|_{t=t_0} = v_0, \quad x_{t=t_0} = x_0.$$

The low of changing of the velocity:

$$v(t) = v_0 + \int_{t_0}^{t} a_0 \, d\tau = v_0 + a_0(t - t_0).$$

To obtain the low for changing of the coordinate we integrate the formula for the velocity:

$$x(t) = x_0 + \int_{t_0}^{t} (v_0 + a_0(\tau - t_0)) \, d\tau.$$

It yields:

$$x(t) = x_0 + v_0(t - t_0) + \frac{a_0}{2}(t - t_0)^2.$$

Observation. Uniformly accelerated moving defines the parabolic dependency of the coordinate on time.

# A periodic movement

Let us consider periodic movement:
$x(t) = a\cos(t+\alpha), \quad a \in \mathbb{R}, \quad \alpha \in [0, 2\pi).$

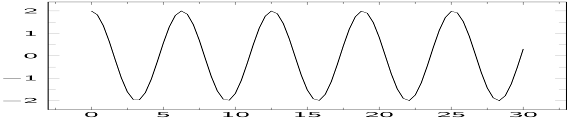



Figure: The example of periodic movement $x = 2\cos(t)$.

In this case the velocity can be written as
$v(t) \equiv \dot{x}(t) = -a\sin(t+\alpha), \quad a \in \mathbb{R}, \quad \alpha \in [0, 2\pi).$

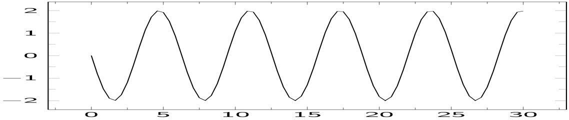
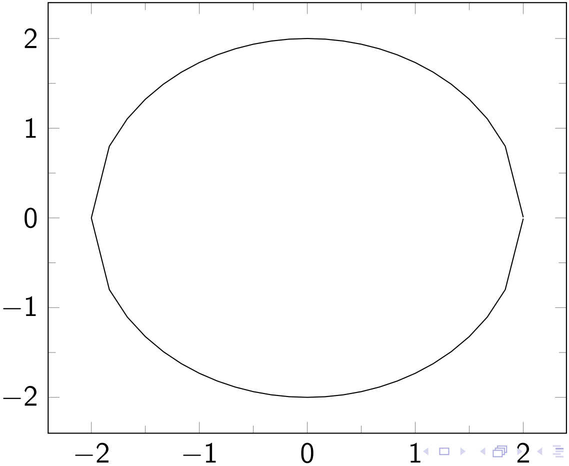



Figure: The example of periodic velocity $v = -2\sin(t)$.

# Phase plane

The trajectory of the particle at the plane $(x(t), v(t))$ is called as the phase trajectory and the plane $(x, v)$ is called as the phase plane. The trajectory for the periodic movement $x = a\cos(x + \alpha)$ and $v = -a\sin(t + \alpha)$ looks as a circle of radius $r = a$ on the phase plane $(x, v)$.



# Periodic movement

Observation.

$$\ddot{x}(t) = -a\cos(t+\alpha) \equiv -x(t).$$

Therefore this periodic movement is defined by the differential equation:

$$\ddot{x} + x = 0.$$

Such equation defines the movement of a load on a spring.

# Decreasing oscillations

The coordinates for decreasing oscillations look like
$x(t) = ae^{-\mu t}\cos(t+\alpha)$ and
$v(t) = -\mu a e^{-\mu t}\cos(t+\alpha) - ae^{-\mu t}\sin(t+\alpha)$.

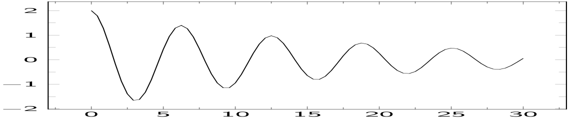



Figure: The example of periodic movement $x = 2\cos(t)$.

Then for polar coordinates $r, \phi$ we obtain:

$$r = \sqrt{x^2+v^2} = \sqrt{1 - \frac{\mu}{a}\sin(2(t+\alpha))}ae^{-\mu t}.$$

Therefore the $r \to 0$ as $t \to \infty$.

$$\tan(\phi) = \frac{v(t)}{x(t)} = \frac{-\sin(t+\alpha) - \mu\cos(t+\alpha)}{\cos(t+\alpha)} = -\tan(t+\alpha) - \mu.$$

# Decreasing oscillations in polar coordinates

For small $\mu$ we obtain: $\phi = -t - \alpha - O(\mu)$.
Hence:$r(\phi) \sim ae^{-\mu(\phi-\alpha)}(1 + O(\mu))$.

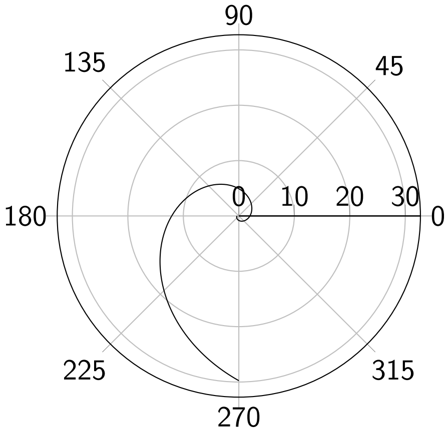



Figure: The decreasing oscillations look as a spiral on the polar coordinate.