Tutorial 10. Non-inertial frames.

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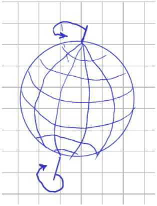
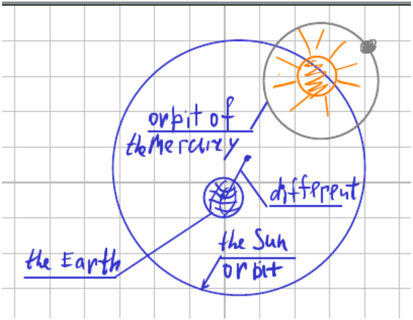
Hipparchus and Ptolemy



- Hipparchus was a Greek astronomer and mathematician (190-120 BC) from Nicaea and Rhodes.Historians argue about Hipparchus invention of the Earth precession. The precession period is about 28000 years.
- Ptolemy was astronomer and mathematician who lived between 100 and 170 years AD in Alexandria. The most famous his work is Almagest. Almagest contains a star catalogue which was partially based on observation during 800 years ancient scientists.

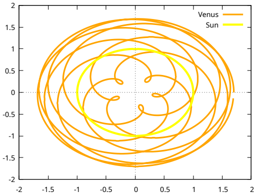
Geocentric planetary motion



Planetary motions were considered as a motion on epicycles over several celestial spheres. The general approach was based on spherical motion as the most ideal shape. Of course, we know that the orbit of the Earth is close to ellipse with small eccentricity. Due to the difference the elliptic motion and the spherical motion the Geocentric approach was needed to a spherical approximations for the observed motion of the Sun and planets.

We will assume that the Sun rotate over the Earth over a cycle. For simplicity consider a circle instead of that ellipse.

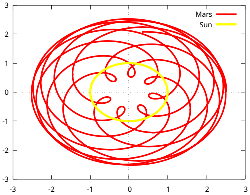
The Sun and Venus



Distance between the Sun and the Earth $R \sim 1$ A.U. and the period of orbital rotation $T_E = 1$. Distance between the Sun and the Venus $R_V \sim 0.7$ A.U. and the period of orbital rotation $T_V = 224/365$. The same assumption we will use for the orbit of the Venus. The approximate formula for motion of the Venus:

$$x = \cos(2\pi t) + R_V \cos\left(2\pi \frac{t}{T_V}\right), \quad y = \sin(2\pi t) + R_V \sin\left(2\pi \frac{t}{T_V}\right).$$

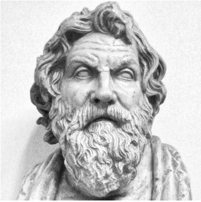
The Sun and Mars



Distance between the Sun and the Earth $R \sim 1$ A.U. and the period of orbital rotation $T_E = 1$. Distance between the Sun and the Mars $R_M \sim 1.524$ A.U. and the period of orbital rotation $T_M = 687/365$. The same assumption we will use for the orbit of the Venus. The approximate formula for motion of the Venus:

$$x = \cos(2\pi t) + R_M \cos\left(2\pi \frac{t}{T_M}\right), \quad y = \sin(2\pi t) + R_M \sin\left(2\pi \frac{t}{T_M}\right).$$

Heliocentric system




- One of the first persons who objective argue of Heliocentric system was an ancient Greek Aristarchus of Samos (470-380 BC). He claim that the Earth revolves over the Sun.
- Also now are known medieval astronomers (like Ulucbek from Samarkand) from Islamic countries who considered the Heliocentric frame as more convenient than the Geocentric approach.

Copernicus and Foucault pendulum

- The most famous and more close to current point of view on this topic was made by N. Copernicus at 1472. He shows that the simplest mathematical forms of the celestial motion look in a heliocentric picture of the Universe.
- At 1851 Foucault applied a pendulum to show a rotation of the Earth.

Non-inertial frames and the Earth rotation

Let us neglect motion of the Solar system around a Milky way kernel. So we consider the Sun as an approximately inertial frame. In this case we can consider two different motions of the Earth.

- The Earth has rotation with a period $T_1 \sim 24 \times 60 \times 60 = 86,400$ sec..
- The Earth radius $R_E \sim 6.4 \times 10^6$ m.
- An equatorial velocity is about

$$V_E = 2\pi \frac{R_E}{T_1} \sim 6.28 \times \frac{6.4 \times 10^6}{86.4 \times 10^3} \sim 464.72 \frac{m}{sec.}.$$

- A centripetal acceleration on the Earth equator is about

$$a_1 \sim (2\pi)^2 \frac{R_E}{T_1^2} \sim (6.28)^2 \times \frac{6.4 \times 10^6}{86.4^2 \times 10^6} \sim 34 \times 10^{-3} \frac{m}{sec^2}.$$

Non-inertial frames and the Earth orbital motion

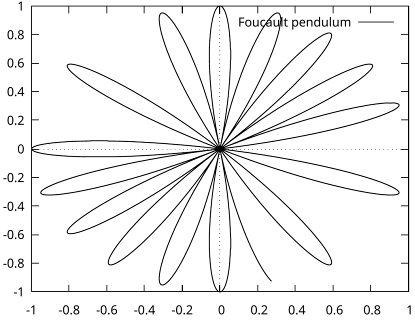
- The Earth has orbital motion with a period about $T_2 \sim 365 \times 24 \times 60 \times 60 = 31,356,000 = 31.356 \times 10^6$ sec..
- The orbital radius $R_O \sim 149.6 \times 10^9$ m.
- An orbital velocity is about

$$V_O = 2\pi \frac{R_O}{T_2} \sim 6.28 \times \frac{149.6 \times 10^9}{31.356 \times 10^6} \sim 30 \times 10^3 \frac{m}{sec}.$$

- A centripetal acceleration on the Earth is about

$$a_2 \sim (2\pi)^2 \frac{R_O}{T_2^2} \sim (6.28)^2 \times \frac{149.6 \times 10^9}{31.356^2 \times 10^{12}} \sim 6 \times 10^{-3} \frac{m}{sec^2}.$$

The Foucault pendulum on the north pole



Let us consider imaginary pendulum right under the axis of rotation of the Earth. An equation for the pendulum looks like follow

$$ml^2\phi'' + mgl\sin(\phi) = 0,$$

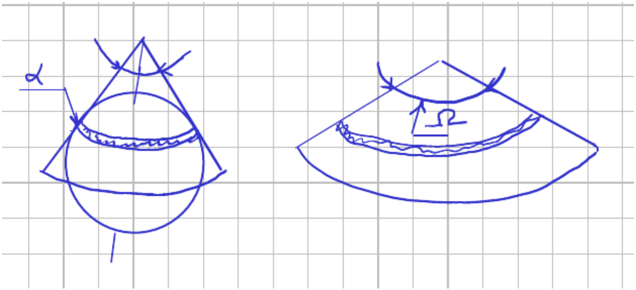
Assume that the angle ϕ is small. It yields:

$$\phi'' + \frac{g}{l}\phi = 0, \quad \phi = a\cos(\omega t + \phi_0),$$

The surface under the pendulum rotate with period
$T = 24 \times 60 \times 60 = 86400$ sec.
The projection of the rotation plain looks like a parametric curve:

$$x(t) = a\cos(\omega t + \phi_0)\cos\left(\frac{2\pi}{T}t\right), \quad y(t) = a\cos(\omega t + \phi_0)\sin\left(\frac{2\pi}{T}t\right).$$

The Foucault pendulum on the average attitudes



A line attitude α can be represented as a cone with an angle on the peak of the development $\Omega = 2\pi \sin(\alpha)$. Therefore to get a monodromy we should be rotate $2\pi/\Omega = 1/\sin(\alpha)$. Then in Innopolis the Foucault pendulum gets one rotation during $24/\sin(55^\circ) \sim 29$ hour and $30'$ min..

Low pressure belts in the Earth

A low pressure on the Earth are the equatorial latitudes and latitudes between 40° and 70° north attitudes.
In this case the cold winds which are anticyclones wind the orient winds and bring a cold from the polar area.
A warm winds which are cyclones wind from west and bring the warm of the tropic area.

Rotating frame

Let an initial position (x, y) is follows:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} r\cos(\alpha) \\ r\sin(\alpha) \end{pmatrix},$$

where $r = \sqrt{x'^2 + y'^2}$.
A turn on the angle ϕ new position is follows:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos(\phi+\alpha) \\ r\sin(\phi+\alpha) \end{pmatrix} = \begin{pmatrix} r(\cos(\phi)\cos(\alpha) - \sin(\phi)\sin(\alpha)) \\ r(\sin(\phi)\cos(\alpha) + \cos(\psi)\sin(\alpha)) \end{pmatrix} =$$
$$\begin{pmatrix} \cos(\phi) r\cos(\alpha) - \sin(\phi) r\sin(\alpha) \\ \sin(\phi) r\cos(\alpha) + \cos(\phi) r\sin(\alpha) \end{pmatrix} = \begin{pmatrix} \cos(\phi)x' - \sin(\phi)y' \\ \sin(\phi)x' + \cos(\phi)y' \end{pmatrix} =$$
$$\begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

Velocity in rotating frame

Assume that $\phi = \omega t$ and $x' = x'(t)$ $y' = y'(t)$. Then:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{d}{dt}\left[\begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix}\right]\begin{pmatrix} x' \\ y' \end{pmatrix} + \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix}\frac{d}{dt}\left[\begin{pmatrix} x' \\ y' \end{pmatrix}\right] = \begin{pmatrix} -\omega\sin(\phi) & -\omega\cos(\phi) \\ \omega\cos(\phi) & -\omega\sin(\phi) \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} + \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix}\frac{d}{dt}\begin{pmatrix} x' \\ y' \end{pmatrix}.$$

Let us show the following:

$$\begin{pmatrix} -\omega\sin(\phi) & -\omega\cos(\phi) \\ \omega\cos(\phi) & -\omega\sin(\phi) \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix}\begin{pmatrix} -\omega y' \\ \omega x' \end{pmatrix}$$

where

$$(-\omega y', \omega x', 0) = \vec{\omega} \times \vec{r}', \quad \vec{\omega} = (0, 0, \omega), \quad \vec{r}' = (x', y', 0).$$

$$\vec{v} = A(\vec{\omega} \times \vec{r}\,' + \vec{v}\,'), \quad \vec{v}\,' = \frac{d}{dt}\vec{r}\,' \quad A = \begin{pmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$\frac{d\vec{v}}{dt} = \frac{d}{dt}[A](\vec{\omega} \times \vec{r}\,' + \vec{v}\,') + A\frac{d}{dt}(\vec{\omega} \times \vec{r}\,' + \vec{v}\,') =$$
$$A\left(\vec{\omega} \times (\vec{\omega} \times \vec{r}\,' + \vec{v}\,') + \frac{d\vec{\omega}}{dt} \times \vec{r}\,' + \vec{\omega} \times \vec{v}\,' + \frac{d\vec{v}\,'}{dt}\right).$$

Hence:

$$\vec{a} = A\left(\vec{\omega} \times (\vec{\omega} \times \vec{r}\,') + \frac{d\vec{\omega}}{dt} \times \vec{r}\,' + 2\vec{\omega} \times \vec{v}\,' + \vec{a}\,'\right).$$

Define external force as $\vec{F} = m\vec{a}$ and $\vec{F}\,' = A^{-1}\vec{F}$, then the forces in the rotational frame are

$$m\vec{a}\,' = \vec{F}\,' - m\frac{d\vec{\omega}}{dt} \times \vec{r}\,' - 2m\vec{\omega} \times \vec{v}\,' - m\vec{\omega} \times (\vec{\omega} \times \vec{r}\,').$$

Coordinates in a non-inertial frame

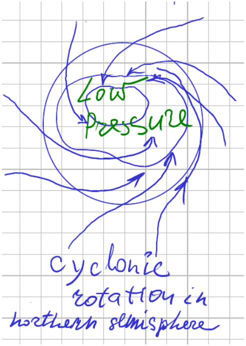
The coordinate vector for an observer in the inertial frame:

$$\vec{r} = \vec{R} + \vec{r}'.$$

Here $\vec{r}$ is coordinate of particle in the inertial frame, $\vec{R}$ is a radius-vector of the center of non-inertial frame and $\vec{r}'$ is a radius vector of the particle with respect to non-inertial frame.

$$m\vec{a}' = \vec{F}' - m\frac{d\vec{\omega}}{dt} \times \vec{r}' - 2m\vec{\omega} \times \vec{v}' - m\vec{\omega} \times (\vec{\omega} \times \vec{r}') + \frac{d^2\vec{R}}{dt^2}.$$

Cyclones and Anticyclones rotation



A consequence of the last formula is a claim that all linear motion in the northern hemisphere due to Coriolis effect $-2\vec{\omega} \times \vec{v}\,'$ deviates from straight direction into right hand side.
Cyclones were defined by low pressure area, therefore the air flow which tends to the center of the cyclones deviates into right-hand side and as a result the air flow rotates anticlockwise.
The anticyclonic air flow are rotated in clockwise direction.