Systems of the first order linear differential equations with constant coefficients

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Systems of first-order equations and linear equations of *n*-th order

$$\frac{d^n y}{dx^n} + \sum_{k=0}^{n-1} a_k \frac{d^k y}{dx^k} = f(x),$$

$$u_1(x) = y(x),\ \ u_2(x) = \frac{dy}{dx}, \ldots,$$

$$u_{k+1}(x) = \frac{d^k y}{dx^k}, \ldots, u_{n-2}(x) = \frac{d^{n-1} y}{dx^{n-1}};$$

$$\frac{du_1}{dx} = u_2,\ \ \frac{du_2}{dx} = u_3, \ldots,$$

$$\frac{du_n}{dx} + \sum_{k=1}^{n-1} a_{k+1} u_k = f(x).$$

$$\frac{d\mathbf{U}}{dx} + \mathbf{AU} = \mathbf{B}(x)$$

Where:

$$\mathbf{U} = \begin{pmatrix} u_1 & u_2 & \dots & u_{n-1} & u_n \end{pmatrix},$$
$$\mathbf{B}(x) = \begin{pmatrix} 0 & 0 & \dots & 0 & f(x) \end{pmatrix}$$

And the coefficients matrix **A** is given by:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ a_1 & a_2 & a_3 & \dots & a_{n-1} & a_n \end{pmatrix}$$

A general form of a system of equations

$$\frac{d\mathbf{Y}}{dx} = \mathbf{A}\mathbf{Y} + \mathbf{B}(x)$$

where:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \ \mathbf{Y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \ \mathbf{B}(x) = \begin{pmatrix} b_1(x) \\ b_2(x) \\ \vdots \\ b_n(x) \end{pmatrix}$$

In this case, the coefficients matrix $\mathbf{A}$ is a constant matrix, and the vector-function $\mathbf{B}(x)$ is a known function of x.

A general form of a system of equations

Therefore, the general form of the linear system of first-order differential equations with a known vector-function on the right-hand side can be represented by:

$$\frac{d\mathbf{Y}}{dx} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} + \begin{pmatrix} b_1(x) \\ b_2(x) \\ \vdots \\ b_n(x) \end{pmatrix}$$

Fundamental system of solutions

To find the eigenvalues and eigenvectors, we can substitute $\mathbf{x} = \mathbf{v}e^{\lambda t}$, where λ is the eigenvalue and $\mathbf{v}$ is the corresponding eigenvector.

$$\frac{d(\mathbf{v}e^{\lambda t})}{dt} = \mathbf{A}(\mathbf{v}e^{\lambda t})$$

We get:

$$\lambda \mathbf{v}e^{\lambda t} = \mathbf{A}\mathbf{v}e^{\lambda t}$$

Dividing both sides by $e^{\lambda t}$ and rearranging, we obtain:

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v}$$

Now, we have a standard eigenvalue-eigenvector equation. To solve for the eigenvalues λ and eigenvectors $\mathbf{v}$, we need to find nontrivial solutions.

Fundamental system of solutions

The eigenvalues λ can be obtained by solving the characteristic equation:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

where $\mathbf{I}$ is the identity matrix of the same size as $\mathbf{A}$.
Once we have the eigenvalues, we can find the corresponding eigenvectors by substituting each eigenvalue back into the equation $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ and solving for $\mathbf{v}$.

Algebraic and geometric dimensions

In a general case the characteristic equation can be rewritten in the form:

$$\prod_{k=1}^{m}(\lambda - \lambda_k)^{m_k} = 0.$$

The λ_k is a root of the characteristic equation of order m_k. The order k is the algebraic repetition of the eigenvalue. Each eigenvalue may correspond to some quantity of eigenvectors. The number of such linear independent vectors is called geometrical order of the eigenvector.

A general case

In a general case the algebraic multiplicity and the geometric one coincide.

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad |A - \lambda I| = (1-\lambda)^2 - 4 = 0,$$

$$\lambda_1 = 3,\ \mathbf{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \lambda_{\mathbf{2}} = -\mathbf{1},\ \mathbf{v_2} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

A fundamental set of the solutions is:

$$\mathbf{y}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3x},\ \mathbf{y}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-x},\ \mathbf{Y} = \begin{pmatrix} e^{3x} & e^{-x} \\ e^{3x} & -e^{-x} \end{pmatrix}.$$

A special example of repeated eigenvalue

Let's consider

$$\frac{d}{dx}\mathbf{y} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathbf{y},$$

then

$$\begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} \equiv (1-\lambda)^2 = 0, \quad \lambda = 1.$$

The eigenvector is a nontrivial solution of the system:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow \mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

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A special example of repeated eigenvalue

Suppose the second solution of the system has a form:

$$\tilde{\mathbf{y}} = xe^{x}\begin{pmatrix}1\\0\end{pmatrix} + e^{x}\tilde{\mathbf{v}}.$$

Substituting the $\tilde{\mathbf{y}}$ one gets:

$$xe^{x}\begin{pmatrix}1\\0\end{pmatrix} + e^{x}\begin{pmatrix}1\\0\end{pmatrix} + e^{x}\tilde{\mathbf{v}} = \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}\left(xe^{x}\begin{pmatrix}1\\0\end{pmatrix} + e^{x}\tilde{\mathbf{v}}\right)$$

$$\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}\tilde{v}_1\\\tilde{v}_2\end{pmatrix} = \begin{pmatrix}\tilde{v}_1+\tilde{v}_2\\\tilde{v}_2\end{pmatrix} \Rightarrow \tilde{v} = \begin{pmatrix}0\\1\end{pmatrix}.$$

A special example of repeated eigenvalue

As a result one gets the fundamental set of solutions might be represented as a matrix:

$$\mathbf{Y} = \begin{pmatrix} e^x & x\, e^x \\ 0 & e^x. \end{pmatrix}.$$

A fundamental system of solutions

Suppose for all λ_k the algebraic and geometric dimensions coincide and is equal to m_k. Then one can get n-th linear independent solutions of given system of differential equations:

$$y_j = \mathbf{v}_{k_1} e^{\lambda_k x},\ y_{j+1} = \mathbf{v}_{k_2} e^{\lambda_k x}, \ldots, y_{j+m} = \mathbf{v}_{k_m} e^{\lambda_k x},$$

where $j = \sum_{i=1}^{k-1} m_i$.

If the algebraic order is m_k and the geometrical dimension of the eigenvectors for the given eigenvalues is j_k, then the additional solutions of the system of equations can be found in the form :

$$\mathbf{y}_{j_k+1} = e^{\lambda_k x}(\mathbf{v}_{j_1} x + \tilde{\mathbf{v}}_{j_1+1}), \ldots, \mathbf{y}_{m_k} = e^{\lambda_k x} \sum_{j=1}^{m_k - j_k} \tilde{\mathbf{v}}_j x^j.$$

Repeated roots of the characteristic equations

To find all the linearly independent eigenvectors corresponding to a repeated eigenvalue λ, we can use the concept of generalized eigenvectors. We find a set of linearly independent generalized eigenvectors associated with λ, which satisfy:

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{v}_j = \mathbf{v}_{j-1}$$

where $\mathbf{v}_j$ represents the jth generalized eigenvector for the eigenvalue λ. Here, j ranges from 1 to m (multiplicity of λ), and we take $\mathbf{v}_0 = \mathbf{0}$.

These generalized eigenvectors span the entire eigenspace associated with λ. If the geometric multiplicity of λ is less than its algebraic multiplicity (i.e., if there are fewer linearly independent eigenvectors than the multiplicity of λ), the remaining linearly independent vectors will be generalized eigenvectors.

Solution of homogeneous system of equations

Let's denote this vector as $\mathbf{y_k}$. Then:

$$\frac{d\mathbf{y_k}}{dx} = \mathbf{A}\mathbf{y_k}$$

To find the solution, we need to find the eigenvalues and eigenvectors of the matrix $\mathbf{A}$. Let's denote an eigenvalue as λ and its corresponding eigenvector as $\mathbf{v}$. For simplicity we will define eigenvalues as $\lambda_i, i \in \{1, \ldots, n\}$, perhaps some of the eigenvalues are equivalent.

The solution of the homogeneous part of the linear system can be expressed as a linear combination of the eigenvectors $\mathbf{v}$ multiplied by exponential terms:

$$\mathbf{x_h} = c_1\mathbf{v}_1 e^{\lambda_1 x} + c_2\mathbf{v}_2 e^{\lambda_2 x} + \ldots + c_n\mathbf{v}_n e^{\lambda_n x}$$

where $c_1, c_2, \ldots, c_n$ are constants determined by initial conditions or additional constraints.

Wronskian of the fundamental set of solutions

Theorem

The Wronskian $W(U)$ of a fundamental set of solutions of the system of the first order differential equations

$$U' = AU$$

is a solution of the equation:

$$W'(U) = \mathrm{tr}(A)W.$$

Here the operator $\mathrm{tr}(A) \equiv \sum_{k=1}^{n} a_{kk}$.

The idea of a proof for the theorem about an evolution of the Wronskian

Let's differentiate the Wronskian and substitute the the right-hand sides of the system of equations:

$$u_k' = \sum_{j=1}^{n} a_{kj} u_j.$$

Then rewrite the sum of n determinants of matrices among which only one term will be linear independent: $a_{kk} u_k$. Pay into attention this properties one gets the statement of the theorem.

The idea of a proof for the theorem about an evolution of the Wronskian

To clarify the idea lets consider the system of two equations:

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} a_{11}u_1 + a_{12}u_2 \\ a_{21}u_1 + a_{22}u_2 \end{pmatrix}.$$

Suppose the set of fundamental solutions is given by the matrix:

$$U = \begin{pmatrix} U_{11}, U_{12} \\ U_{12}, U_{22} \end{pmatrix},\ W(U) = U_{11}U_{22} - U_{21}U_{12},$$

$$W' = U_{11}'U_{22} + U_{11}U_{22}' - U_{21}'U_{12} - U_{21}U_{12}' =$$

$$(a_{11}U_{11} + a_{12}U_{21})U_{22} + U_{11}(a_{21}U_{12} + a_{22}U_{22}) -$$

$$(a_{21}U_{11} + a_{22}U_{21})U_{12} - U_{21}(a_{11}U_{12} + a_{12}U_{22})$$

$$= (a_{11} + a_{22})(U_{11}U_{22} - U_{12}U_{21}) = \mathrm{tr}(A)W.$$

Non-homogeneous systems

Let's consider the system:

$$Y' = AY + B.$$

Define the fundamental set of solutions for the complimentary system (homogeneous one):

$$U' = AU, \quad \det(U) \neq 0.$$

Denote $Y = U \cdot C(x)$, where $C(x)$ is vector of unknown functions. After substitution of the formula for Y into the equation one gets:

$$U' \cdot C + U \cdot C' = A \cdot U \cdot C + B,$$
$$U \cdot C' + U' \cdot C - A \cdot U \cdot C = B,$$
$$U \cdot C' + (U' - A \cdot U) \cdot C = B.$$

Non-homogeneous systems

Through the non-zero value of the Wronskian for the fundamental set of solutions the inverse matrix of U exists and hence:

$$U \cdot C' = B \Rightarrow C' = U^{-1}B,$$
$$C = \int U^{-1}(x) \cdot B(x)dx.$$

Non-homogeneous system. An example

$$\frac{d}{dx}\mathbf{y} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathbf{y} + \begin{pmatrix} \sin(x) \\ 1 \end{pmatrix}.$$

$$U = \begin{pmatrix} e^x & x\,e^x \\ 0 & e^x. \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} e^{-x} & -x\,e^{-x} \\ 0 & e^{-x}. \end{pmatrix},$$

$$y = \begin{pmatrix} e^x & x\,e^x \\ 0 & e^x. \end{pmatrix} \int \begin{pmatrix} e^{-x} & -x\,e^{-x} \\ 0 & e^{-x}. \end{pmatrix} \begin{pmatrix} \sin(x) \\ 1 \end{pmatrix} dx =$$

$$\begin{pmatrix} e^x & x\,e^x \\ 0 & e^x. \end{pmatrix} \begin{pmatrix} \int e^{-x}\sin(x) - xe^{-x}dx \\ \int e^{-x}dx \end{pmatrix}.$$

Non-homogeneous system. An example

$$y = \begin{pmatrix} e^x & x\,e^x \\ 0 & e^x. \end{pmatrix} \begin{pmatrix} -e^{-x}\left(\frac{1}{2}(\sin(x)+\cos(x)) - (x+1)\right) \\ -e^{-x} \end{pmatrix} + \begin{pmatrix} e^x & x\,e^x \\ 0 & e^x \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix},$$

$$y = \begin{pmatrix} 1 - \frac{1}{2}(\sin(x)+\cos(x)) \\ -1 \end{pmatrix} + \begin{pmatrix} e^x C_1 + x\,e^x C_2 \\ e^x C_2 \end{pmatrix}.$$

Example a pendulum

The equation for the pendulum has the form:

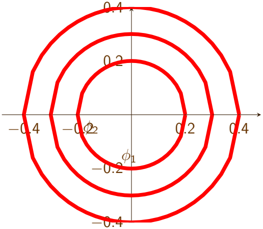
$$\ddot{\phi} + \sin(\phi) = 0.$$

Present this equation in form of a system of the first-order equation:

$$\begin{pmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{pmatrix} = \begin{pmatrix} \phi_2 \\ -\sin(\phi_1) \end{pmatrix}.$$

Obviously, the pendulum has two points of equilibrium. There are $(\phi_1, \phi_2) = (0, 0)$ and $(\phi_1, phi_2) = (\pi, 0)$. For studying properties around these points we linearize the equation. That means we remain the linear part of the equation only.

An example. A pendulum

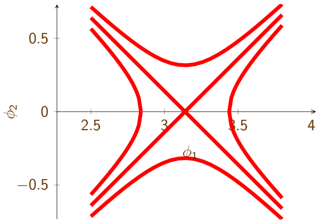


At
the point $(\phi_1, \phi_2) = (0, 0)$ we get:

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ -y_1 \end{pmatrix}.$$

Here $y_1 \sim \phi_1$ and $y_2 \sim \phi_2$. Then due to the classifications of the system of two differential equations of the first order one get the center at the point $(y_1, y_2) = (0, 0)$.

Example a pendulum

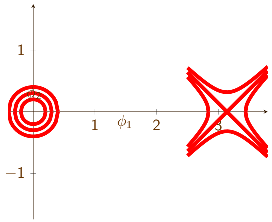


At
the point $(\phi_1, \phi_2) = (\pi, 0)$ we get

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ y_1 \end{pmatrix}.$$

Here $y_1 \sim \pi - \phi_1$ and $y_2 \sim \phi_2$.
Then we get the saddle
point at the point $(y_1, y_2) = (0, 0)$

Example a pendulum



Yet another example. A predator-pray system

A.J. Lotka (1925) and V. Volterra (1926) assumed the model with population of two kind like predators and preys.
Let x be number of preys and y be numbers of predators. The preys reproduced proportional their quantity and disappear proportional the numbers of the predators:

$$dx = (\alpha_1 x - \beta_1 yx)dt, \quad \alpha_1, \beta_1 > 0.$$

The number of the predators increases proportional by the preys and disappeared proportional their quantity:

$$dy = (-\alpha_2 y + \beta_2 yx)dt, \quad \alpha_2, \beta_2 > 0.$$

Yet another example. A predator-pray system

As a result the system of the differential equations are:

$$\frac{dx}{dt} = (\alpha_1 - \beta_1 y)x,$$
$$\frac{dy}{dt} = -(\alpha_2 - \beta_2 x)y.$$

The simplest form of the predator-prey model

The points of equilibrium are $(x, y) = (0, 0)$ and $(x, y) = (\alpha_2/\beta_2, \alpha_1/\beta_1)$.
It is convenient to change the variables:

$$x = \frac{\alpha_2}{\beta_2} u, \quad y = \frac{\alpha_1}{\beta_1} v.$$

As a result we obtain:

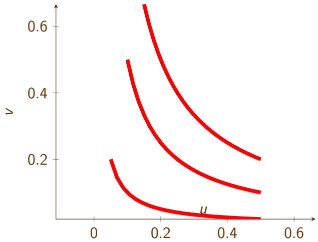
$$\frac{du}{dt} = a_1(1 - v)x, \quad \frac{dv}{dt} = -a_2(1 - u)v.$$

The changing of the independent variable $t = \tau/a_1$ yields:

$$\frac{du}{d\tau} = (1 - v)u, \quad \frac{dv}{d\tau} = -k(1 - u)v.$$

Here $k = a_2/a_1$ is a parameter of the model.

The neighborhoods of equilibrium points of the predator-prey model

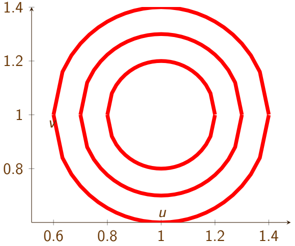


In the neighborhood of the origin the linearized system looks like:

$$\frac{du}{d\tau} \sim u, \quad \frac{dv}{d\tau} \sim -kv.$$

So the solutions are $u \sim u_0 \exp(\tau)$ and $v \sim v_0 \exp(-k\tau)$.
Therefore the point $(0, 0)$ is a saddle for the linearized equation.

The neighborhoods of equilibrium points of the predator-prey model



The linear equation in the neighborhood of the point $(u, v) = (1, 1)$ can be obtained after the changing of the variables:

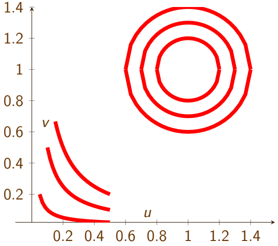
$$u = X + 1, \quad v = Y + 1.$$

The linear system for X, Y has the form:

$$\frac{dX}{d\tau} = Y, \quad \frac{dY}{d\tau} = -kX.$$

According to the previous classification we obtain the center for the linear approximation around the equilibrium $(1, 1)$.

The neighborhoods of equilibrium points of the predator-prey model



Summary