Linear differential equations with constant coefficients.
Special non-homogeneous cases and boundary value problems

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An equation with an exponential external force

$$u'' + u = e^{kx}.$$

Let's assume a particular solution of the form $u_p(x) = Ae^{kx}$, where A is a constant to be determined.
Now, substitute this assumed solution into the differential equation:

$$u_p'' + u_p = (Ae^{kx})'' + Ae^{kx}$$

Taking the derivatives, we have:

$$u_p'' + u_p = Ak^2e^{kx} + Ae^{kx}$$

Substituting into equation, we get:

$$(Ak^2 + A)e^{kx} = e^{kx}.$$

An example

Comparing this with the right-hand side of the original differential equation, e^{kx}, we can equate the corresponding terms:

$$Ak^2 + A = 1, \quad A = \frac{1}{k^2+1}.$$

Therefore, the particular solution is:

$$u_p(x) = \frac{e^{kx}}{k^2+1}$$

A general solution looks as follows:

$$u(x) = u_p(x) + u_c(x) = \frac{e^{kx}}{k^2+1} + C_1 \cos(x) + C_2 \sin(x)$$

where C_1 and C_2 are arbitrary constants.

The observation

The main rule

Special non-homogeneous parts involve a certain solution using method of undetermined coefficients:

$$y^{(n)} + \sum_{k=0}^{n-1} a_k y^{(k)} = e^{\lambda x}.$$

$$y_p = \frac{e^{\lambda x}}{\lambda^n + \sum_{k=0}^{n-1} a_k \lambda^k}$$

for the case: $\lambda^n + \sum_{k=0}^{n-1} a_k \lambda^k \neq 0$.

The case of monomial with exponent factor

$$y^{(n)} + \sum_{k=0}^{n-1} a_k y^{(k)} = b_m x^m e^{\lambda x}$$

An observation:

n-th order derivatives of the term $b_m x^m e^{\lambda x}$ are monomials of the same power.

A method of undefined coefficients.

A particular solution can be found in the following form:

$$y(x) = \sum_{l=0}^{m} c_l x^l e^{\lambda x}, \ \forall \lambda : \ \ \lambda^n + \sum_{k=0}^{n-1} a_k \lambda^k \neq 0.$$

Receipt.

1. Check if $\lambda^n + \sum_{k=0}^{n-1} a_k \lambda^k \neq 0$.
2. Substitute the formula into the equation.
3. Gather coefficients of linear independent monomials $x^l e^{\lambda x}$.
4. Equate all coefficients to zero and obtain a system of linear equations for $c_l, \quad l \in \{0, 1, \dots, m\}$.
5. Solve a system of linear equations for $c_l, \quad l \in \{0, 1, \dots, m\}$.

A particular solution of $u'' + u = x^2e^{2x}$

1. The characteristic equation for complementary equation is $r^2 + 1 = 0$,. The $2^2 + 1 \neq 0$.
2. Substitute $y_p(x) = c_2x^2e^{2x} + c_1xe^{2x} + c_0e^{2x}$ into the equation:
   $5c_2x^2e^{2x} + 8c_2xe^{2x} + 5c_1xe^{2x} + 2c_2e^{2x} + 4c_1e^{2x} + 5c_0e^{2x} = x^2e^{2x}$.
3. Gather coefficients:
   $(5c_2 - 1)x^2e^{2x} + (8c_2 + 5c_1)xe^{2x} + (c_2 + 4c_1 + 5c_0)e^{2x} = 0$.
4. Equate the coefficients to zero:
   $5c_2 - 1 = 0,\ 8c_2 + 5c_1 = 0,\ c_2 + 4c_1 + 5c_0 = 0$.
5. Solve the system: $c_2 = \frac{1}{5},\ c_1 = -\frac{8}{25},\ c_0 = \frac{22}{125}$.

$$y_p = \frac{1}{5}x^2e^{2x} - \frac{8}{25}xe^{2x} + \frac{22}{125}e^{2x}.$$

A trigonometric case

Consider an equation:

$$y^{(n)} + \sum_{k=0}^{n-1} a_k y^{(k)} = (b_m x^m \cos(\omega x) + d x^j \sin(\omega x)) e^{\lambda x}.$$

An observation:

To construct a particular solution for this equation the general form of solution should be as follows:

$$y_p = \sum_{l=0}^{M} c_l x^l e^{\lambda x} \cos(\omega x) + \sum_{l=0}^{M} s_l x^l e^{\lambda x} \sin(\omega x),$$

$$\forall \lambda + i\omega : \quad (\lambda + i\omega)^n + \sum_{k=0}^{n-1} a_k (\lambda + i\omega)^k \neq 0, \; M = \max\{m, j\}.$$

A special form of the non-homogeneous part

Consider an equation

$$u'' - 4u = e^{2x}.$$

To find a particular solution one can use the method of variations of parameters:

$$y_p = c_1(x)e^{2x} + c_2(x)e^{-2x},$$
$$c_1'e^{2x} + c_2'e^{-2x} = 0, \quad 2c_1'e^{2x} - 2c_2e^{-2x} = e^{2x}.$$
$$4c_1'e^{2x} = e^{2x}, \quad 4c_2'e^{-2x} = -e^{2x};$$
$$c_1 = \frac{x}{4}, \quad c_2 = frac116e^{4x};$$
$$y_p = \frac{x}{4}e^{2x} - \frac{1}{16}e^{2x}.$$

A rule for constructing solution

An observation:

If the non-homogeneous part of the linear equation with constant coefficients contains a monomial x^m and exponent $e^{\lambda x}$, where $\lambda^n + \sum_{k=0}^{n-1} a_k \lambda^k = 0$ and λ is a root of multiplicity j, then the partial solution solution has a form:

$$y_p = \sum_{k=j}^{m+j} b_k x^k e^{\lambda x}.$$

The coefficients b_k can be found using the method of undetermined coefficients.

A resonant case

$$y'' + \omega^2 y = \cos(\omega x)$$

Let's construct a particular solution using a method of variations of parameters:

$$y_p = a(x)\cos(\omega x) + b(x)\sin(\omega x);$$

$$\begin{cases} a'\cos(\omega x) + b'\sin(\omega x) = 0, \\ -a'\omega\sin(\omega x) + b'\omega\cos(\omega x) = \cos(\omega x); \end{cases}$$

$$a' = -\frac{1}{\omega}\cos(\omega x)\sin(\omega x), \;\; b' = \frac{1}{\omega}\cos^2(\omega x) = \frac{1}{\omega}\left(\frac{1}{2} + \cos(2\omega x)\right);$$

$$a = \frac{1}{2\omega^2}\cos(2\omega x), \quad b = \frac{x}{2\omega} + \frac{1}{2\omega^2}\sin(2\omega x);$$

$$y_p = \frac{x}{2\omega}\sin(\omega x) - \frac{1}{2\omega^2}\cos(\omega x)\cos(2\omega x) + \frac{1}{2\omega^2}\sin(\omega x)\sin(2\omega x).$$

A rule of constructing solution

An observation:

If the non-homogeneous part of the linear equation with constant coefficients contains terms $(ax^m\cos(\omega x)+bx^l\sin(\omega x))e^{\lambda x}$, where $(\lambda+i\omega)^n+\sum_{k=0}^{n-1}a_k(\lambda+i\omega)^k=0$ and $\lambda+i\omega$ is a root of multiplicity j, then the partial solution solution has a form:

$$y_p=\sum_{k=j}^{M+j}(c_kx^k\cos(\omega x)+s_kx^k\sin(\omega x))e^{\lambda x},\ \ M=\max\{m,j\}.$$

The coefficients c_k and s_k can be found using the method of undetermined coefficients.

A superposition principle

If $y_1(x)$ is a solution of the equation

$$y^{(n)} + \sum_{k=0}^{n-1} a_k y^{(k)} = \mathbf{f_1(x)}$$

and $y_2(x)$ is a solution of the equation

$$y^{(n)} + \sum_{k=0}^{n-1} a_k y^{(k)} = \mathbf{f_2(x)}$$

then $Y(x) = y_1(x) + y_2(x)$ is a solution of the following equation:

$$y^{(n)} + \sum_{k=0}^{n-1} a_k y^{(k)} = \mathbf{f_1(x)} + \mathbf{f_2(x)}.$$

This statement can be checked by substitution.

The resonance in the second-order equation

Resonance in second-order equations occurs when the system is excited at a frequency that matches the natural frequency of the system.

$$\ddot{x} + \mu\dot{x} + \omega^2 x = F\sin(\Omega t),$$

where μ is the damping coefficient, ω is the natural frequency of the solution and F is the amplitude of external force.
Let's define: $\tau = \omega t,\ \eta = \frac{\mu}{\omega},\ \kappa = \frac{\Omega}{\omega},\ u = \frac{\omega^2}{F}x$, as a result we get:

$$\ddot{u} + \eta\dot{u} + u = \sin(\kappa\tau),$$

The resonance in the second-order equation

Assume $\eta \neq 0$ and $\eta^2 < 4$.
The particular solution

$$x_p = \frac{-\eta}{(1-\kappa^2)^2 + \eta^2\kappa^2} \cos(\kappa t) + \frac{(1-\kappa^2)}{(1-\kappa^2)^2 + \eta^2\kappa^2} \sin(\kappa t).$$

Define $A = \frac{1}{\sqrt{(1-\kappa)^2+\eta^2\kappa^2}}$, rewrite the particular solution:

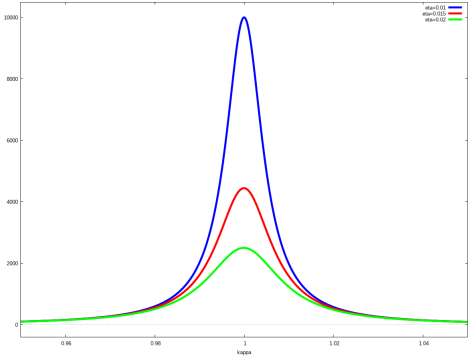
$$x_p(t) = A(\cos(\kappa\tau)\sin(\phi) + \sin(\kappa\tau)\cos(\phi)),$$

$$\sin(\phi) = -\frac{\eta\kappa}{\sqrt{(1-\kappa^2)^2 + \eta^2\kappa^2}},$$

$$\cos(\phi) = \frac{(1-\kappa^2)F}{\sqrt{(1-\kappa^2)^2 + \eta^2\kappa^2}},$$

$$\tan(\phi) = -\frac{\eta\kappa}{(1-\kappa^2)^2},$$

The resonance in the second-order equation



$$u_p = \frac{\sin(\kappa\tau + \phi)}{\sqrt{(1-\kappa^2)^2 + \eta^2\kappa^2}}, \quad .$$

An example of boundary value problem

$$y'' + 4y' + 4y = 0, \quad y(0) = 1, \quad y(\pi) = 0$$

The characteristic equation is $r^2 + 4r + 4 = 0$, which has a repeated root of $r = -2$. Therefore, the general solution is of the form

$$y(x) = (c_1 + c_2 x)e^{-2x}$$

Due to the boundary conditions we get:

$$y(0) = c_1 = 1, \quad y(\pi) = (c_1 + c_2\pi)e^{-2\pi} = 0,$$

$$c_1 + c_2\pi = 0 \Rightarrow c_2 = -\frac{1}{\pi}.$$

Therefore, the solution to the boundary value problem is

$$y(x) = \left(1 - \frac{x}{\pi}\right) e^{-2x}$$

Yet another boundary value problem

$$y'' + y = 0,$$
$$y(0) = 0, \quad y(\pi) = 1.$$

The general solution is of the form

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$

To find the values of c_1 and c_2, we use the boundary conditions:

$$y(0) \equiv c_1 \cos(0) + c_2 \sin(0) = 0 \Rightarrow c_1 = 0,$$
$$y(\pi) \equiv c_1 \cos(0) + c_2 \sin(0) = 1 \Rightarrow 0 = 1$$

Therefore, the boundary value problem does not have solution.

General approach for solving boundary value problem for homogeneous equation

Let's consider the problem:

$$y'' + a_1 y' + a_0 y = 0,$$
$$\alpha_0 y(x_0) + \beta_0 y'(x_0) = A,$$
$$\alpha_1 y(x_1) + \beta_1 y'(x_1) = B.$$

Here $\alpha_0, \alpha_1, \beta_0, \beta_1$ and A, B are constants. Let's general solution of the equation has the form:

$$y = c_1 y_1(x) + c_2 y_2(x).$$

Then the equations for defining the coefficients c_1 and c_2:

$$\alpha_0(c_1 y_1(x_0) + c_2 y_2(x_0)) + \beta_0(c_1 y_1'(x_0) + c_2 y_2'(x_0)) = A,$$
$$\alpha_1(c_1 y_1(x_1) + c_2 y_2(x_2)) + \beta_1(c_1 y_1'(x_1) + c_2 y_2'(x_2)) = B.$$

Condition for existence of solution for the boundary value problem

Rewrite the system of equations for c_1 and c_2:

$$(\alpha_0 y_1(x_0) + \beta_0 y_1'(x_0))c_1 + (\alpha_0 y_2(x_0) + \beta_0 y_2'(x_0))c_2 = A,$$
$$(\alpha_1 y_1(x_1) + \beta_1 y_1'(x_1))c_1 + (\alpha_1 y_2(x_1) + \beta_1 y_2'(x_1))c_2 = B.$$

The solvability of the boundary value problem is determined by the solvability of the system of linear equations.

Eigenvalue problem for the second-order differential equation

The eigenvalue problem is defined as the problem to find the values of λ and non-trivial solutions for the homogeneous boundary values:

$$\begin{aligned} y'' + a_1 y' + a_0 y &= \lambda y, \\ \alpha_0 y(x_0) + \beta_0 y'(x_0) &= 0, \\ \alpha_1 y(x_1) + \beta_1 y'(x_1) &= 0. \end{aligned}$$

The appropriate values of λ are called eigenvalues and related solutions $y(x)$ are called as eigenfunctions.

An example

Consider the differential equation:

$$\frac{d^2y}{dx^2} = \lambda y,\ y(0) = 0,\ y(\pi) = 0.$$

Let $\lambda = k^2 > 0$ then:

$$\begin{aligned} y &= c_1 e^{kx} + c_2 e^{-kx}, \\ c_1 + c_2 = 0, \quad c_1 e^k + c_2 e^{-k} &= 0 \Rightarrow \\ c_1 = 0,\ c_2 &= 0. \end{aligned}$$

Hence, the eigevalues goes not exist as $\lambda > 0$.

An example

Let $\lambda = 0$ then:

$$\begin{aligned} y &= c_1 + c_2 x, \\ c_1 = 0, \quad c_2\pi &= 0 \Rightarrow \\ c_1 &= 0, \; c_2 = 0. \end{aligned}$$

Hence, $\lambda = 0$ does not an eigenvalue.

An example

Let $\lambda = -\omega^2 < 0$, then

$$y(x) = c_1 \cos(\omega x) + c_2 \sin(\omega x)$$

Using the boundary conditions, we can determine the values of c_1:

$$c_1 = 0, \quad c_2 \sin(\omega \pi) = 0 \Rightarrow \\ \omega \in \mathbb{N}.$$

The eigenvalues for given problem $\lambda = -n^2, \; n \in \ltimes$ and corresponding eigenfunctions are:

$$y_n(x) = \sin(nx).$$

A Green's function

Let's consider non-homogeneous equation

$$y'' + a_1 y' + a_0 y = f(x)$$

and zero boundary conditions $y(x_0) = 0,\ y(x_1) = 0$. Suppose we construct two linear independent solutions of the homogeneous equation and certain boundary conditions:

$$y_1(x_0) = 0,\ y_1(x_1) \neq 0,$$
$$y_2(x_0) \neq 0,\ y_2(x_1) = 0.$$

Let $w(y_1, y_2)$ be a Wronskian of the functions $y_1(x$ and $y_2(x)$, then the function

$$y(x) = y_2(x) \int_{x_0}^{x} \frac{y_1(\xi)}{w(y_1, y_2)} f(\xi) d\xi + y_1(x) \int_{x}^{x_1} \frac{y_2(\xi)}{w(y_1, y_2)} f(\xi) d\xi$$

is the solution of given boundary problem.

A Green's function

Indeed:

$$y' = y_2(x)' \int_{x_0}^{x} \frac{y_1(\xi)}{w(y_1, y_2)} f(\xi) d\xi + y_1(x)' \int_{x}^{x_1} \frac{y_2(\xi)}{w(y_1, y_2)} f(\xi) d\xi,$$

$$y'' = y_2''(x) \int_{x_0}^{x} \frac{y_1(\xi)}{w(y_1, y_2)} f(\xi) d\xi +$$

$$y_1''(x) \int_{x}^{x_1} \frac{y_2(\xi)}{w(y_1, y_2)} f(\xi) d\xi + f(x).$$

Substituting into the equation gives an equality.

A Green's function

The function

$$G(x,\xi) = \begin{cases} y_2(x)\frac{y_1(\xi)}{w(y_1,y_2)}, & x_0 < \xi < x; \\ y_1(x)\frac{y_2(\xi)}{w(y_1,y_2)}, & x < \xi < x_1. \end{cases}$$

is called **Green's function.**

The solution of the boundary valued problem for the given equation can be written as follows:

$$y = \int_{x_0}^{x_1} G(x,\xi)f(\xi)d\xi.$$

An example

Consider the differential equation:

$$\frac{d^2y}{dx^2} - \lambda y = f(x),\ y(0) = 0,\ y(\pi) = 0.$$

Let $\lambda = k^2 > 0$ then:

$$y_1 = \frac{1}{2}(e^{kx} - e^{-kx}) = \sinh(kx),$$

$$y_2 = \frac{1}{2}(e^{k(x-\pi)} - e^{-k(x-\pi)}) = \sinh(k(x-\pi)).$$

$$y(x) = \int_0^\pi G(x, \xi f(\xi) d\xi,$$

$$G(x,\xi) = \begin{cases} \sinh(k(x-\pi))\frac{\sinh(\xi)}{w(y_1,y_2)},\ 0 < \xi < x; \\ \sinh(x)\frac{\sinh(k(\xi-\pi))}{w(y_1,y_2)},\ x < \xi < \pi. \end{cases}$$

$$w(y_1, y_2) = k\sinh(-k\pi).$$

An example

Let $\lambda = 0$ then:

$$y_1 = x, \; y_2 = x - \pi;$$
$$w(y_1, y_2) = x - (x - \pi) = \pi;$$
$$y(x) = \int_0^\pi G(x, \xi f(\xi) d\xi,$$
$$G(x,\xi) = \begin{cases} (x-\pi)\frac{\xi}{\pi}, \; 0 < \xi < x; \\ x\frac{(\xi-\pi)}{\pi}, \; x < \xi < \pi. \end{cases}.$$

An example

Let $\lambda = -\omega^2 < 0, \ \ \omega \notin \mathbb{Z}$ then

$$y_1(x) = \sin(\omega x), \ \ y_2 = \sin(\omega(x-\pi)),$$
$$w(y_1, y_2) = \omega \sin(\pi\omega),$$
$$G(x,\xi) = \begin{cases} \sin(\omega(x-\pi))\frac{\sin(\omega\xi)}{\omega\sin(\pi\omega)}, \ \ 0 < \xi < x; \\ \sin(\omega x)\frac{\sin(\omega(\xi-\pi))}{\omega\sin(\pi\omega)}, \ \ x < \xi < \pi. \end{cases}.$$
$$y(x) = \int_0^\pi G(x,\xi) f(\xi) d\xi.$$

Summary