High-order linear differential equations

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High-order linear homogeneous equation with constant coefficients

The general form of a high-order differential equation with constant coefficients is:

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

where y is the dependent variable and $a_{n-1}, \cdots, a_1, a_0$ are constants.

Try to find a solution using a substitution $y = e^{rx}$

As a result we get a polynomial equation in r, which is called the characteristic equation.

$$r^n + \sum_{k=0}^{n-1} a_k r^k = 0$$

Fundamental theorem of algebra

Let $a_0, a_1, \ldots, a_{n-1}$ be real numbers. Then the polynomial equation

$$r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0 = 0$$

has exactly n complex roots, counting multiplicities.
The polynomial can be represented as follows:

$$P(r) = (r - r_1)^{k_1} \ldots (r - r_i)^{k_i} \ldots (r - \rho_j)^{k_j}(r - \bar{\rho}_j)^{k_j}, \ \sum_l k_l = n.$$

The roots of the characteristic equation determine the form of the solution.

Examples

$$y'' + a_1 y' + a_0 y = 0,$$

where y is the dependent variable and a_2, a_1, a_0 are constants. To find the characteristic polynomial, we assume a solution of the form $y = e^{rx}$ and substitute it into the differential equation.

$$r^2 e^{rx} + a_1 r e^{rx} + a_0 e^{rx} = 0$$

Dividing both sides by e^{rx}, we get:

$$r^2 + a_1 r + a_0 = 0$$

This is the characteristic equation, and its roots determine the form of the solution.

Real roots of the characteristic equation

For example, the differential equation:

$$y'' + y' - 2y = 0$$

has the characteristic polynomial:

$$r^2 + r - 2 = 0$$

which has the roots $r_1 = 1$ and $r_2 = -2$. Therefore, the general solution of the differential equation is:

$$y = c_1 e^x + c_2 e^{-2x}$$

Complex roots of the characteristic equation

2. Complex roots: If the roots of the characteristic equation are complex conjugates, for example, the differential equation:

$$y'' + 2y' + 2y = 0$$

has the characteristic polynomial:

$$r^2 + 2r + 2 = 0$$

which has the roots $r = -1 + i$ and $r = -1 - i$. Therefore, the general solution of the differential equation is:

$$\begin{aligned} y = C_1 e^{(-1+i)x} + C_2 e^{(-1-i)x} = \\ C_1 e^{-x}(\cos(x) + i\sin(x)) + C_2 e^{-x}(\cos(x) - i\sin(x)) = \\ e^{-x}\left((C_1 + C_2)\cos(x) + i(C_1 - C_2)\sin(x)\right). \end{aligned}$$

Complex roots of the characteristic equation

Let $C_1 = p + iq$, $C_2 = s + it$:

$$y = e^{-x}\left((p+s+i(q+t))\cos(x) + i((p-s+i(q-t))\sin(x)\right) = \\ e^{-1}((p+s)\cos(x) + (t-q)\sin(x)) + \\ ie^{-x}((q+t)\cos(x) + (p-s)\sin(x)).$$

If $p = s$ and $t = -q$ then the solution is real-valued function:

$$y = e^{-x}(a\cos x + b\sin x),$$

where $a = 2p$ and $b = 2t$. In other words to obtain a real-valued solution one can $C_1 = p + iq$ and $C_2 = p - q \Rightarrow C_2 = \overline{C}_1$.

Complex roots of the characteristic equation

General real solution for complex conjugated roots is given

$$y = e^{\alpha x}(a\cos(\beta x) + b\sin(\beta x)).$$

Repeated roots of the characteristic equation

For example, the differential equation:

$$y'' + 4y' + 4y = 0$$

has the characteristic polynomial:

$$r^2 + 4r + 4 = 0,$$

which has the repeated root $r = -2$. Therefore, the general solution of the differential equation is:

$$y = c_1 e^{-2x} + c_2 x e^{-2x}$$

The n-order differential equations

There are three cases depending on the roots of the characteristic equation:

1. Real root r_j of counting k_j define k_j numbers of solutions of the ordinary differential equation:

$$\{y_j(x)\}_{j=1}^{k_j} = \{e^{r_j x},\ xe^{r_j x}, \dots, x^{k_j-1}e^{r_j x}\}$$

2. The complex conjugated roots $\rho_m = \alpha_m \pm i\beta_m$ of the characteristic equation imply the set of the solutions in the forms:

$$\{y_m(x)\}_{j=1}^{k_m} = \{y_m(x) = e^{\alpha_m x}(a_1 \cos(\beta_m x) + b_1 \sin(\beta_m x)), \dots, \\ x^{k_m-1}e^{\alpha_m x}(a_m \cos(\beta_m x) + b_m \sin(\beta_m x))\}.$$

The n-order differential equations

Theorem

Let $y_1(x)$ and $y_2(x)$ be solutions of the equation

$$y^{(n)} + \sum_{k=0}^{n-1} a_k y^{(k)} = 0,$$

then

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

is the solution of given equation for any $C_1, C_2 \in \mathbb{R}^2$.

Proof. This theorem can be proved by substituting into the equation.

linearly independent functions

A set of functions $y_1(x), y_2(x), \ldots, y_n(x)$ is linearly independent if and only if the only solution to the equation

$$c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) = 0$$

for all x in the domain of the functions is the trivial solution $c_1 = c_2 = \cdots = c_n = 0$.

linearly independent functions

To define a system of equations for the $c_k, k \in \{1, \dots, n\}$ differentiate the formula for linear combination by $(n-1)$ times. It yields:

$$\begin{pmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \dots & \dots & \dots & \dots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \dots \\ c_n \end{pmatrix} = 0.$$

This equation has non-trivial solution if determinant of the matrix is equal to 0.

Wronskian

The determinant of the matrix:

$$W \equiv \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \dots & \dots & \dots & \dots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix}$$

is called Wronskian.

Theorem

If the Wronskian does not equal to 0 at certain point on the interval of discontinuity of $y_k \, \forall k \in 1, 2, \dots, n$ then this Wronskian does not equal to 0 on whole interval of discontinuity of $y_k \, \forall k \in 1, 2, \dots, n$.

Proof of the theorem

Let's differentiate the Wronskian:

$$\frac{d}{dx}W = \begin{vmatrix} y_1' & y_2' & \dots & y_n' \\ y_1' & y_2' & \dots & y_n' \\ \dots & \dots & \dots & \dots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix} + \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1'' & y_2'' & \dots & y_n'' \\ \dots & \dots & \dots & \dots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix} + \dots +$$

$$\begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \dots & \dots & \dots & \dots \\ y_1^{(n)} & y_2^{(n)} & \dots & y_n^{(n)} \end{vmatrix} = -a_{n-1} \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \dots & \dots & \dots & \dots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix}.$$

As a result we get:

$$\frac{d}{dx}W = -a_{n-1}W, \quad W = W_0 e^{-a_{n-1}x}.$$

An example

For example, the differential equation

$$y'' + 4y' + 4y = 0$$

has the fundamental set of solutions $\{e^{-2x}, xe^{-2x}\}$. The Wronskian is follows:

$$W = \begin{vmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & e^{-2x} - 2xe^{-2x} \end{vmatrix} =$$

$$-2e^{-2x}xe^{-2x} - e^{-2x}e^{-2x} - e^{-2x}(-2x)e^{-2x} = -e^{-4x}.$$

General solution

Definition of fundamental set of solutions

A set of linearly independent solutions $\{y_k(x)\}_{k=1}^{n}$ of a homogeneous n-th order linear differential equation is called a *fundamental set of solutions* for give equation.

A genera solution of a homogeneous n-th order linear differential equation can be written in the form:

$$y(x) = \sum_{k=1}^{n} C_k y_k(x), \;\; C_k \in \mathbb{R}.$$

An initial value problem. An example

For the equation

$$y'' + y' - 2y = 0$$

we have found the fundamental set of equations: $\{e^x, e^{-2x}\}$.
Let's find a solution of initial valueproblem:

$$y_{x=0} = 3,\ \ y'|_{x=0} = 0.$$

Then the system of equations for C_1 and C_2 is follows:

$$\begin{aligned} C_1 + C_2 = 3, \quad C_1 - 2C_2 = 0 \Rightarrow C_1 = 2C_2,\ 3C_2 = 3 \Rightarrow \\ C_2 = 1,\ C_1 = 2 \Rightarrow \\ y = 2e^x + e^{-2x}. \end{aligned}$$

Solution of an initial value problem

Let the initial value problem be written as follows:

$$y|_{x=x_0} = Y_0,\ y'|_{x=x_0} = Y_1, \ldots y^{(n-1)}|_{x=x_0} = Y_{n-1}.$$

Then substituting the general solution into the left-hand sides of the initial data one gets:

$$\begin{aligned} \sum_{k=1}^{n} C_k y_k|_{x=x_0} &= Y_0, \\ \sum_{k=1}^{n} C_k y_k'|_{x=x_0} &= Y_1, \\ &\ldots \\ \sum_{k=1}^{n} C_k y_k^{(n-1)}|_{x=x_0} &= Y_{n-1}. \end{aligned}$$

Solution of an initial value problem

Using a matrix form one gets:

$$\left.\begin{pmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \dots & \dots & \dots & \dots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{pmatrix}\right|_{x=x_0} \begin{pmatrix} C_1 \\ C_2 \\ \dots \\ C_n \end{pmatrix} = \begin{pmatrix} Y_0 \\ Y_1 \\ \dots \\ Y_{n-1} \end{pmatrix}.$$

The determinant of the matrix in the left-hand side is Wronski's determinant (Wronskian). This determinant for the fundamental set of solutions is not equal zero. Then the matrix has an inverse matrix and hence the system of linear equations for unknown constants $C_1, \dots, C_n$ has an unique non-trivial solution.

An example

Consider the equation

$$y'' + y = f(x)$$

and try to construct a certain solution using a method of variations of parameters.
The general real valued solution of the homogeneous differential equation is:

$$y = c_1 \cos(x) + c_2 \sin(x).$$

An example

The certain solution will construct in the form:

$$y_* = c_1(x)\cos(x) + c_2(x)\sin(x).$$

Then

$$\begin{aligned} y_*' &= c_1'\cos(x) - c_1\sin(x) + c_2'\sin(x) + c_2\cos(x), \\ y_*'' &= c_1''\cos(x) - 2c_1'\sin(x) - c_1\cos(x) + \\ &\quad c_2''\sin(x) + 2c_2'\cos(x) - c_2\sin(x), \end{aligned}$$

An example

Substitute the formulas for y_* into the origin equation:

$$c_1'' \cos(x) - 2c_1' \sin(x) - c_1 \cos(x) +$$
$$c_2'' \sin(x) + 2c_2' \cos(x) - c_2 \sin(x) +$$
$$c_1(x) \cos(x) + c_2(x) \sin(x) = \tan(x).$$

Then:

$$c_1'' \cos(x) - 2c_1' \sin(x) + c_2'' \sin(x) + 2c_2' \cos(x) = f(x).$$

As a result we obtain one equation with two unknown functions, which are $c_1(x)$ and $c_2(x)$.

An example

Eventually we need to find only one function. So we can add additional equation. For simplicity we consider an additional condition:

$$c_1' \cos(x) + c_2' \sin(x) = 0.$$

This condition implies a reduction of the second derivatives of c_1 and c_2 form the final equation:

$$\begin{aligned} y_*' &= -c_1 \sin(x) + c_2 \cos(x) + (c_1' \cos(x) + c_2' \sin(x)), \\ y_*'' &= -c_1' \sin(x) - c_1 \cos(x) + c_2' \cos(x) - c_2 \sin(x), \end{aligned}$$

Substituting we get:

$$\begin{aligned} -c_1' \sin(x) - c_1 \cos(x) + c_2' \cos(x) - c_2 \sin(x) \\ + c_1 \cos(x) + c_2 \sin(x) = f(x). \end{aligned}$$

An example

At a result we obtain system of two equations for c_1' and c_2':

$$c_1' \cos(x) + c_2' \sin(x) = 0, \quad -c_1' \sin(x) + c_2' \cos(x) = f(x).$$

Then

$$c_1' = -f(x)\sin(x), \quad c_2' = f(x)\cos(x).$$

The final formula for the certain solution is follows:

$$y_*(x) = -\cos(x)\int_{x_0}^{x} f(\xi)\sin(\xi)d\xi + \sin(x)\int_{x_0}^{x} f(\xi)\cos(\xi)d\xi,$$

where x_0 might be any constant fro a range of the function $f(x)$.

A non-homogeneous equation

Let $f(x)$ be known function, consider a non-homogeneous equation:

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x)$$

Let's try to find a certain solution of the equation using a method of variation of parameters. Suppose a form of the certain solution is follows:

$$y_*(x) = \sum_{k=1}^{n} C_k(x) y_k(x), \;\; C_k(x) \in \mathbf{C}^1.$$

Here $C_1(x), \ldots, C_n(x)$ are unknown functions.

A non-homogeneous equation

We must mention that the number of unknown functions is redundant because we only need to find one function eventually. Therefore, we will exploit this condition..

Our goal is to substitute the formula for y_* into the origin differential equation. Beforehand, we define the derivatives of y_*. Let's differentiate the expression of y_*.

$$\frac{d}{dx}y_*(x) = \sum_{k=1}^{n} C_k(x)y_k'(x) + \sum_{k=1}^{n} C_k'(x)y_k(x).$$

To simplify the expression for the second derivative and to avoid an appearance of high-order derivatives of unknown $C_k(x), k \in \{1, \ldots, n\}$ we suppose:

$$\sum_{k=1}^{n} C_k'(x)y_k(x) = 0.$$

A non-homogeneous equation

The second derivative of $y_*(x)$ is follows:

$$\frac{d^2}{dx^2}y_*(x) = \sum_{k=1}^{n} C_k(x)y_k''(x) + \sum_{k=1}^{n} C_k'(x)y_k'(x).$$

The same trick we use for simplify the formula for the second derivative:

$$\sum_{k=1}^{n} C_k'(x)y_k'(x) = 0.$$

As a result it yields:

$$\frac{d^2}{dx^2}y_*(x) = \sum_{k=1}^{n} C_k(x)y_k''(x).$$

A non-homogeneous equation

Following to the same way for j-th derivative ($j < n$) we get:

$$\frac{d^j}{dx^j} y_*(x) = \sum_{k=1}^{n} C_k(x) y_k^{(j)}(x) + \sum_{k=1}^{n} C_k'(x) y_k^{(j-1)}(x)$$

and add the equation:

$$\sum_{k=1}^{n} C_k'(x) y_k^{(j-1)}(x) = 0.$$

As a result we get:

$$\frac{d^j}{dx^j} y_*(x) = \sum_{k=1}^{n} C_k(x) y_k^{(j)}(x).$$

A non-homogeneous equation

Eventually we obtain a formula fro the n-th-order derivative

$$\frac{d^n}{dx^n}y_*(x) = \sum_{k=1}^{n} C_k(x)y_k^{(n)}(x) + \sum_{k=1}^{n} C_k'(x)y_k^{(n-1)}(x)$$

and $n-1$ additional equations in the form:

$$\sum_{k=1}^{n} C_k'(x)y_k^{(j-1)}(x) = 0,\ j \in \{1, \ldots, n-1\}.$$

A non-homogeneous equation

Substitute all formulas for the derivatives into the origin equation and collect all coefficients with any factor C_k. Using the property for $\{y_k\}_{k=1}^n$ to be of solution for homogeneous equation one obtains:

$$\sum_{k=1}^{n} C_k'(x) y_k^{(n-1)}(x) = f(x).$$

A non-homogeneous equation

As a result we get:

$$\sum_{k=1}^{n} C_k'(x) y_k^{(j-1)}(x) = 0,\ j \in \{1, \ldots, n-1\},$$

$$\sum_{k=1}^{n} C_k'(x) y_k^{(n-1)}(x) = f(x).$$

Rewrite the same in the matrix form:

$$\begin{pmatrix} y_1 & y_2 & \ldots & y_n \\ y_1' & y_2' & \ldots & y_n' \\ \ldots & \ldots & \ldots & \ldots \\ y_1^{(n-1)} & y_2^{(n-1)} & \ldots & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} C_1' \\ C_2' \\ \ldots \\ C_n' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \ldots \\ f(x) \end{pmatrix}.$$

A non-homogeneous equation

Due to the property of fundamental set the matrix can be inverted and the final formula for the unknown column of $C(x)$'s can be written as follows:

$$\begin{pmatrix} C_1' \\ C_2' \\ \dots \\ C_n' \end{pmatrix} = \begin{pmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \dots & \dots & \dots & \dots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ \dots \\ f(x) \end{pmatrix}.$$

The right hand sides of the formula are known and one can integrate both sides of the formulas on x. As a result one obtains the functions $C_k(x), k \in \{1, 2, \dots, n\}$ and finally construct the certain solution:

$$y_*(x) = \sum_{k=1}^{n} C_k(x) y_k(x).$$

Summary

Definitions

Independent solutions

An initial value problem

Non-homogeneous equation