Solutions in form of series

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Outlines

A linear first-order example

Let's consider a linear equation

$$u' = u.$$

Suppose that there exists a solution as a convergent series:

$$u = \sum_{n=0}^{\infty} u_n x^n,$$

Suppose that one can differentiate the series by terms and the result is a convergent series for the derivative. Then:

$$\sum_{n=0}^{\infty} n u_n x^{n-1} = \sum_{n=0}^{\infty} u_n x^n,$$

A linear first-order example

Since the linearly independence of the set of polynomial x^n one can equate coefficients of x^n. As a result one get:

$$u_n = \frac{u_{n-1}}{n}, \quad u_n = \frac{u_0}{n!},$$

$$u(x) = u_0 \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

$$u(x) = u_0 e^x.$$

Due to separability of the space of continuously differentiable functions one can claim that the constructed solution is unique.

A non-linear first-order example

$$u' = u^2, \quad u = \sum_{n=0}^{\infty} u_n x^n,$$

$$\sum_{n=0}^{\infty} n u_n x^{n-1} = \left(\sum_{n=0}^{\infty} u_n x^n \right)^2,$$

The right-hand side can be written though a convolution:

$$\left(\sum_{n=0}^{\infty} u_n x^n \right)^2 = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} u_k u_{n-k} \right) x^n.$$

A non-linear first-order example

$$\sum_{n=1}^{\infty} n u_n x^{n-1} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} u_k u_{n-k} \right) x^n.$$

A substitution allows to find coefficients of the series.

$$\begin{aligned} u_1 &= u_0^2, \\ 2u_2 &= u_0 u_1 + u_1 u_0, \quad u_2 = u_0 u_1, \quad u_2 = u_0^3; \\ 3u_3 &= u_0 u_2 + u_1^2 + u_2 u_0, \quad u_3 = \frac{1}{3}(2u_0 u_2 + u_1^2), \quad u_3 = u_0^4. \end{aligned}$$

A non-linear first-order example

As a result one gets:

$$u(x) \;=\; \sum_{n=0}^{\infty} u_0^{n+1} x^n,$$

If $|x| < |u_0|$, then the series converges and

$$u(x) \;=\; \frac{u_0}{1 - u_0 x}.$$

However, we know the solution:

$$u(x) = \frac{u_0}{1 - u_0 x}, \quad \forall x \in \mathbb{R}.$$

Theoretical basements for constructing of the solutions like a series are following:

- a convergent series for the solution;
- a change of the limits like a termwise (term by term) differentiation of the series;
- a linear independence of the polynomials;
- a separability of the functional space.

Theorem about differentiation of the series

If a series is convergent uniformly to a function $u(x)$ at $[a, b]$ and the term-by-term derivative of the series is convergent uniformly, then the term-by-term differentiated series converges to the derivative $u'(x)$:

$$u'(x) = \left(\sum_{n=0}^{\infty} u_n x^n \right)' = \sum_{n=1}^{\infty} n\, u_n x^{n-1}.$$

Idea of the proof of the theorem

Let's consider an integral:

$$\int_a^x \left(\sum_{n=1}^{N} n\, u_n \xi^{n-1} + \varepsilon \right) d\xi = \sum_{n=1}^{N} \int_a^x n\, u_n \xi^{n-1} d\xi \, + (x-a)\varepsilon =$$

$$\sum_{n=1}^{N} u_n x^n - \sum_{n=1}^{N} u_n a^n + (x-a)\varepsilon = u(x) - u(a) + \epsilon.$$

Here

$$\epsilon = \left(\sum_{n=1}^{N} u_n x^n - u(x) \right) - \left(\sum_{n=1}^{N} u_n a^n - u(a) \right) + (x-a)\varepsilon.$$

One can show as $N \to \infty$ $\varepsilon \to 0$ and hence $\epsilon \to 0$.

A counterexample for the term by term differentiation

The following series converges at $x = 1$;

$$f(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n},$$

but the term by term derivative of this series

$$f'(x) = \sum_{n=0}^{\infty} (-1)^n x^n$$

does not converge at $x = 1$.

A counterexample for the term by term differentiation

However, the $f(x)$ is the Taylor series for the function

$$f(x) = \log(1 + x),$$

and the derivative exists at $x = 1$:

$$\frac{d}{dx}\log(1 + x) = \frac{1}{1 + x}|_{x=1} = \frac{1}{2}.$$

The linear independence of the polynomials

Let's consider a set of the polynomials x^n, $n \in \mathbb{N}$.

$$\sum_{n=0}^{N} \alpha_n x^n \equiv 0, \quad \sum_{n=0}^{N} \alpha_n^2 \neq 0, \quad \forall x \in [a, b].$$

Theorem about the separability of the differential functions

The Weierstrass theorem

If a function $f(x)$ is continuous on $[a, b]$, then there exists a sequence of polynomials $\{P_n(x)\}_{n=0}^{\infty}$ which converges to $f(x)$. Another words:

$$\forall \epsilon > 0 \ \exists \{P_n(x)\}_{n=0}^{\infty}, \exists N(\epsilon), \forall n > N : \\ |f(x) - P_n(x)| < \epsilon, \ x \in [a, b].$$

This means there exists a countable everywhere dense sequence on the set of continuous functions. Then this set is separable by the definition of the separable functional spaces.

A proof of the Weierstrass theorem

Without loss of generality let's consider $f(x),\ x \in [0,1],\ f(0) = f(1) = 0,\ f(x) \neq 0, \forall x \notin (0,1)$ and a sequence of the polynomials

$$Q_n(x) = q_n(1-x^2)^n,\ n \in \mathbb{N}, \quad \int_{-1}^{1} Q_n(x)dx = 1,$$

$$\int_{-1}^{1}(1-x^2)^n dx \geq 2\int_0^1 (1-x^2)^n dx \geq 2\int_0^{1/\sqrt{n}} (1-x^2)^n dx \geq$$

$$2\int_0^{1/\sqrt{n}} (1-nx^2)dx = \frac{4}{3\sqrt{n}} \geq \frac{1}{\sqrt{n}}, \quad q_n < \sqrt{n}.$$

Let's define

$$P_n(x) = \int_{-1}^{1} f(x+\xi)Q_n(\xi)d\xi \equiv \int_0^1 f(\zeta)Q_n(\zeta - x)d\zeta,\ \zeta = x + \xi.$$

A proof of the Weierstrass theorem

Note:

$$|f(x) - f(x+\xi)| < \frac{\varepsilon}{2},\ |\xi| < \delta(\varepsilon), \quad max|f(x)| = A,$$

$$|f(x) - P_n(x)| = |\int_{-1}^{1}(f(x) - f(x+\xi))Q_n(\xi))d\xi| \leq$$

$$\int_{-1}^{1}|f(x) - f(x+\xi)|Q_n(\xi)d\xi \leq$$

$$\int_{-1}^{-\delta}|f(x) - f(x+\xi)|Q_n(\xi))d\xi +$$

$$\int_{-\delta}^{\delta}|f(x) - f(x+\xi)|Q_n(\xi)d\xi +$$

$$\int_{\delta}^{1}|f(x) - f(x+\xi)|Q_n(\xi)d\xi$$

A proof of the Weierstrass theorem

$$\leq \frac{\epsilon}{2} 2\delta(\epsilon) + 4A \int_{\delta}^{1} Q_n(\xi) d\xi \leq$$

$$\frac{\varepsilon}{2} + 4A\sqrt{n}(1-\delta^2)^n,$$

$\forall \varepsilon > 0, \ \exists N, \forall n > N : (1-\delta^2)^n < \frac{\varepsilon}{2}$, then:

$$|f(x) - P_n(x)| < \varepsilon.$$

Benefits and imperfections

- A constructive procedure for building of the solution.
- A local usable formulas for solutions.
- A convergence for the linear equation does not depend on an initial value.
- Convergence of the series depends on the initial data for the non-linear equation.

An example

Let's consider the same approach for a second order equation:

$$u'' + u = 0,$$

Here the solution will construct in the series form:

$$u = \sum_{n=0}^{\infty} u_n x^n.$$

Assume that second derivative can be obtained by termwide differential of the series. Then one gets:

$$\sum_{n=0}^{\infty} (n+1)(n+2) u_{n+2} x^n + \sum_{n=0}^{\infty} u_n x^n = 0,$$

An example

Due to the independence of the polynomials the equation is equivalent to the sequence of the equalities:

$$u_{n+2} = -\frac{u_n}{(n+1)(n+2)}$$

These equalities can be represented as follows:

$$u_{2n} = (-1)^n \frac{u_0}{(2n)!}, \; u_{2n+1} = (-1)^n \frac{u_1}{(2n+1)!}, \quad n \in \mathbb{N}.$$

Here u_0 and u_1 are parameters of the solution.

An example

As a result one gets:

$$u = u_0 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + u_1 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

$$u(x) = u_0 \cos(x) + u_1 \sin(x).$$

So, we obtain the result by straightforward calculations.

A general case for the second-order equation

$$y'' + a(x)y' + b(x)y = 0,$$
$$a(x) = \sum_{n=0}^{\infty} a_n x^n, \quad b(x) = \sum_{n=0}^{\infty} b_n x^n.$$

Suppose there exists a solution in the form:

$$y(z) = \sum_{n=0}^{\infty} y_n x^n.$$

Substitute the the formula into the equation:

$$\sum_{n=2}^{\infty} n(n-1)y_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n \sum_{n=1}^{\infty} n y_n x^n + \sum_{n=0}^{\infty} b_n x^n \sum_{n=0}^{\infty} y_n x^n = 0.$$

A general case for the second-order equation

As a result we get a sequence of equations for the coefficients y_n:

$$
\begin{aligned}
2y_2 + a_0 y_1 + b_0 y_0 = 0, \\
y_2 = -\frac{1}{2}(a_0 y_1 + b_0 y_0), \\
6y_3 + 2a_0 y_2 + a_1 y_1 + b_0 y_1 + b_1 y_0 = 0, \\
y_3 = -\frac{1}{6}(2a_2 y_2 + a_1 y_1 + b_0 y_1 + b_1 y_0), \\
\dots, \\
(n+1)(n+2)y_{n+2} + \sum_{k=0}^{n} a_{n-k}(k+1)y_{k+1} + \sum_{k=0}^{n} b_{n-k} y_k = 0.
\end{aligned}
$$

$$
y_{n+2} = \frac{-1}{(n+1)(n+2)} \sum_{k=0}^{n} \left(a_{n-k}(k+1)y_{k+1} + b_{n-k} y_k\right)
$$

The Airy equation

$$y'' - xy = 0,$$

$$y_{n+2} = \frac{y_{n-1}}{(n+2)(n+1)} \Rightarrow y_{n+3} = \frac{y_n}{(n+3)(n+2)},$$

$$y_{3n} = y_0 \frac{1 \cdot 4 \cdot \dots \cdot (3n-2)}{(3n)!}, \; y_{3n+1} = y_1 \frac{2 \cdot 5 \cdot \dots \cdot (3n-1)}{(3n+1)!},$$

$$y_{3n+2} = 0.$$

$$y(x) = y_0 \sum_{n=1}^{\infty} \frac{1 \cdot 4 \cdot \dots \cdot (3n-2)}{(3n)!} +$$

$$y_1 \sum_{n=1}^{\infty} \frac{2 \cdot 5 \cdot \dots \cdot (3n-1)}{(3n+1)!}.$$

A power series for Bessel function

Let's consider the Bessel equation

$$y'' + \frac{1}{x}y' + \left(1 - \frac{\nu^2}{x^2}\right) y = 0.$$

The equation has singularity at the origin and the previous approach does not apply.
We assume that the solution can be expressed as a power series of the form:

$$y(x) = x^{\alpha} \sum_{n=0}^{\infty} a_n x^n$$

where a_n are the coefficients to be determined and α is a constant exponent.

A power series for Bessel function

Differentiating $y(x)$ with respect to x gives:

$$\frac{dy}{dx} = \alpha x^{\alpha-1}\sum_{n=0}^{\infty} a_n x^n + x^\alpha \sum_{n=0}^{\infty} n a_{n-1} x^n$$

$$\frac{d^2y}{dx^2} = \alpha(\alpha-1)x^{\alpha-2}\sum_{n=0}^{\infty} a_n x^n + 2\alpha x^{\alpha-1}\sum_{n=0}^{\infty} n a_{n-1} x^n + x^\alpha \sum_{n=0}^{\infty} n(n-1) a_{n-2} x^n.$$

A power series for Bessel function

Substitute the formulas into the Bessel equation:

$$\alpha(\alpha-1)x^{\alpha-2}\sum_{n=0}^{\infty} a_n x^n + 2\alpha x^{\alpha-1}\sum_{n=0}^{\infty} n a_{n-1} x^n$$

$$+x^{\alpha}\sum_{n=0}^{\infty} n(n-1)a_{n-2}x^n +$$

$$\frac{1}{x}\left(\alpha x^{\alpha-1}\sum_{n=0}^{\infty} a_n x^n + x^{\alpha}\sum_{n=0}^{\infty} n a_{n-1} x^n\right) +$$

$$\left(1-\frac{\nu}{x^2}\right)x^{\alpha}\sum_{n=0}^{\infty} a_n x^n = 0$$

A power series for Bessel function

Gather terms of the same order wit respect to power of x:

$$x^{-2}\left(\alpha(\alpha-1)+\alpha-\nu^2\right)\sum_{n=0}^{\infty}a_nx^n+\sum_{n=0}^{\infty}n(n-1)a_{n-2}x^n+\frac{1}{x}\left((2\alpha+1)\sum_{n=0}^{\infty}na_{n-1}x^n\right)+\sum_{n=0}^{\infty}a_nx^n=0.$$

Then

$$\alpha^2-\nu^2=0\Rightarrow\alpha=\pm\nu.$$

A power series for Bessel function

As a result we obtain: and we obtain an equality:

$$\sum_{n=0}^{\infty} n(n-1)a_{n-2}x^n + \frac{1}{x}\left((2\nu+1)\sum_{n=0}^{\infty} na_{n-1}x^n\right) + \sum_{n=0}^{\infty} a_n x^n = 0.$$

Then

$$x^{-1}(2\alpha+1)a_1 = 0 \Rightarrow a_1 = 0,$$

$$x^0((2\alpha+1)2a_2 + a_0 + 2a_2) = 0 \Rightarrow a_2 = \frac{a_0}{2\alpha+4},$$

$$x^1((2\alpha+1)3a_3 + a_1 + 2\cdot 3a_3) = 0 \Rightarrow a_3 = \frac{a_1}{6\alpha+9},$$

$$x^n((2\alpha+1)(n+2)a_{n+2} + a_n + (n+1)\cdot(n+2)a_{n+2}) = 0 \Rightarrow$$

$$a_{n+2} = \frac{-a_n}{(2\alpha+2+n)(n+2)}.$$

A power series for Bessel function

Then any odd coefficients equal zero. Then we might choose $n = 2k$ and the power series for solution of the the Bessel equation has the form:

$$y(x) = Cx^{\nu} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{x^{2k}}{2\nu \cdot (2\alpha + 2) \ldots (2\nu + 2 + 2k)}.$$

Regular singularities

Definition

If the equation has the form:

$$y'' + \frac{a(x)}{(x - x_0)} y' + \frac{b(x)}{(x - x_0)^2} y = 0$$

and the functions $a(x), b(x)$ have Taylor series at the point x_0, the point x_0 is called regular singular point of the equation.

General rule for the regular singularity

The statement

In a neighborhood of the regular singularity point x_0 a solution of the differential equation can be represented in the form:

$$y(x) = x^{\alpha} \sum_{n=0}^{\infty} (x - x_0)^n y_n,$$

where α is a solution of the indicial equation:

$$\alpha(\alpha - 1) + a(x_0)\alpha + b(x_0) = 0.$$

Summary

Examples. First-order equations

Theoretical basements

Second-order equations with regular coefficients

Regular singularities