Envelopes and irregular points

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A parabola of safety

Suppose one uses the tennis ball machine into indoor tennis court.

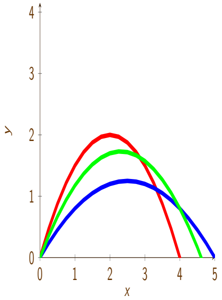
A problem

Where should one install video cameras to avoid a damage ones.

The answer

Find a formula for a parabola of safety!

A parabola of safety



Assume $v_0 = \text{const}$ and $\alpha \in (0, \pi/2)$ is a parameter of the problem.

$$\dot{x} = v_0 \cos(\alpha),\ \ddot{y} = -g.$$

$$x\big|_{t=0} = 0,\ y\big|_{t=0} = 0,\ \dot{y}\big|_{t=0} = v_0 \sin(\alpha).$$

$$x(t) = v_0 t \cos(\alpha),\ \dot{y} = v_0 \sin(\alpha) t - gt,$$

$$y(t) = v_0 \sin(\alpha) t - g\frac{t^2}{2}.$$

So, the parametric formula for the trajectory can be written as:

$$x(t) = v_0 t \cos(\alpha),\ y(t) = v_0 \sin(\alpha) t - g\frac{t^2}{2}.$$

A parabola of safety

Rewrite:

$$t = \frac{x}{v_0 \cos(\alpha)},\ \alpha \neq \frac{\pi}{2}.$$

Then:

$$y = \tan(\alpha)x + \frac{g}{2v_0^2 \cos^2(\alpha)}x^2.$$

Denote:

$$C = \tan(\alpha), \quad a = \frac{g}{2v_0^2}.$$

Here $C \in \mathbb{R}$ is a parameter of the problem and $a = \text{const}$ because we cannot change gravity acceleration g and the initial speed v_0.

So the family of parabolas looks as following:

$$y = Cx + a(1 + C^2)x^2, \quad C \in \mathbb{R}.$$

The parabola of safety

Let's derive a differential equation for the family of parabolas:

$$y = Cx + a(1 + C^2)x^2, \quad C \in \mathbb{R}.$$

Differentiate on x:

$$\frac{dy}{dx} = C + a(1 + C^2)x$$

Find the value of C for given (x, y):

$$C = \pm \frac{\sqrt{-2ay - a^2x^2 + 1} + 1}{ax}.$$

The parabola of safety

Substitute the formula for C into the differential equation:

$$\frac{dy}{dx} = \frac{\pm\sqrt{-2ay - a^2 x^2 - 2ay + 1} + 1}{ax}$$

An implicit form looks more convenient:

$$\left(ax\frac{dy}{dx} - 1\right)^2 = -2ay - a^2 x^2 - 2ay + 1.$$

A failure of an uniqueness condition

Let' check the the uniqueness condition by differentiating the right-hand side of the explicit form of the differential equation:

$$F(x,y) = \frac{\pm\sqrt{-2ay - a^2\,x^2 - 2ay + 1} + 1}{ax},$$

$$\frac{dF}{dy} = \frac{2\sqrt{-2ay - a^2x^2 + 1} - 1}{x\sqrt{-2ay - a^2x^2 + 1}}.$$

So, the condition fails at the curve:

$$y = \frac{1}{2a} - \frac{a}{2}x^2.$$

This equation defines the envelope curve for the family of the parabolas.

Differential equation for given family of curves

Let's consider one parametric family of curves:

$$\Phi(x, y, C) = 0, \quad C \in A \subset \mathbb{R}.$$
$$\forall C \in A, \exists y = y(x, C) \Leftrightarrow x = x(y, C), \quad (x, y) \in \mathbb{R}^2.$$

The aim is to derive the differential equations for the family of curves.

Reasons to find the differential equation

Suppose one designs a manipulator, liking as a robotic arm, for a coffee.

- One knows every trajectory of motion for such robotic arm.
- Problem for this equipment is to limit of acceleration of motion to avoid spilling the drink.
- To solve the problem one should find the dependency of derivative with respect to coordinates and define the dangerous area for limiting the acceleration.

A derivation of the equation

To derive the differential equation one should change formula for the family of curves to excluding the dependency of parameter C.

The receipt for removing the parameter

- Differentiate on x (or on y) the equation of the curves family, to obtain addition equation, which contains the derivative $\frac{dy}{dx}$ (or $\frac{dx}{dy}$).
- Derive the dependency of C using one of the equations.
- Substitute the obtained formula for C into another equation.

Derivation of the equation for the family of curves

$$\Phi(x, t, C) = 0.$$

Differentiate the equation on x or on y:

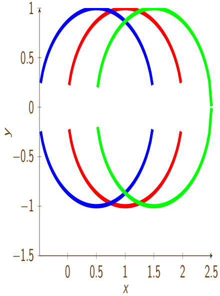
$$\frac{\partial \Phi(x, y, C)}{\partial y} \frac{dy}{dx} + \frac{\partial \Phi(x, y, C)}{\partial x} = 0$$

the same form with derivative on y:

$$\frac{\partial \Phi(x, y, C)}{\partial y} + \frac{\partial \Phi(x, y, C)}{\partial x} \frac{dx}{dy} = 0.$$

Both form are equivalent we can choose. So one can choose more convenient form for further calculations.

An example



A family of the curves:

$$(x - C)^2 + y^2 - 1 = 0.$$

Let's derive the equation for the family.

$$2yy' + 2(x - C) = 0,\ C = yy' + x, \Rightarrow$$
$$(x - (yy' + x))^2 + y^2 = 1,$$

As
a result the equation can be written as:

$$(yy')^2 + y^2 = 1.$$

An envelope

There exists a curve which touch every curve from the family.

Definition

We will call an *envelope curve* such curve that every one point of this curve touches one and only one of the curves of given family.

An envelope

So the envelope function touch different curves of the family then the parameter $C = C(x, y)$. In this case the equation for the envelope function:

$$\frac{d}{dx}\Phi(x, y, C(x, y)) = 0,$$

or

$$\frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial y}\frac{dy}{dx} + \frac{\partial \Phi}{\partial C}\left(\frac{\partial C}{\partial x} + \frac{\partial C}{\partial y}\frac{dy}{dx}\right) = 0.$$

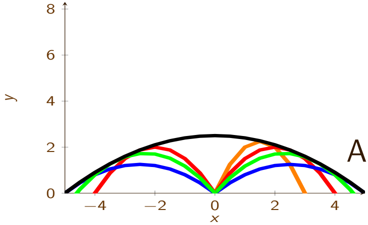
Here

$$\frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial y}\frac{dy}{dx} = 0, \quad \frac{\partial C}{\partial x} + \frac{\partial C}{\partial y}\frac{dy}{dx} \not\equiv 0.$$

So the equations for the envelope are:

$$\frac{\partial \Phi(x, y, C)}{\partial C} = 0, \quad \Phi(x, y, C) = 0.$$

An example



The parabola of safety:

$$y - Cx - a(1 + C^2)x^2 = 0,$$
$$-x - 2aCx^2 = 0$$

As a result:

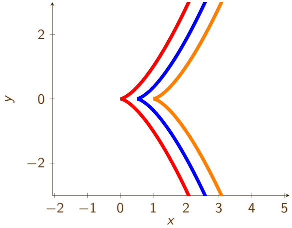
$$C = -\frac{1}{2ax},$$

$$y + \frac{1}{2ax}x - a\left(1 + \left(\frac{1}{2ax}\right)^2\right)x^2 = 0.$$

Finally

$$y + \frac{1}{4a} - ax^2 = 0.$$

Irregular points. An example



Let's consider
a family of semi-qubic parabolas:

$$(x - C)^3 - y^2 = 0.$$

All curves of the family
has splitting points $(x, y) = (C, 0)$.
Define the differential equation for
the family:

$$\frac{dy}{dx} = \frac{3(x - C)^2}{2y}.$$

The right-hand side of the equation at the splitting points
looks as a fraction $\frac{0}{0}$.

Irregular points

Definition

The irregular points of the family of curves $\Phi(x, y, C) = 0$ are defined by the equations:

$$\frac{\partial\Phi(x, y, C}{\partial x} = 0, \quad \frac{\partial\Phi(x, y, C)}{\partial y} = 0.$$

In geometrical point of view this means that the differential equation does not defined in such points:

$$\frac{dy}{dx} = \frac{\frac{\partial\Phi(x,y,C)}{\partial y}}{\frac{\partial\Phi(x,y,C)}{\partial x}}.$$

The set of irregular points

Let's consider the equation for the set of irregular points of the family of curves:

$$\frac{\partial \Phi(x,y,C)}{\partial x}+\frac{\partial \Phi(x,y,C)}{\partial y}\frac{dy}{dx}+\frac{\partial \Phi(x,y,C)}{\partial C}\left(\frac{\partial C}{\partial x}+\frac{\partial C}{\partial y}\frac{dy}{dx}\right)=0$$

In the irregular points:

$$\frac{\partial \Phi(x,y,C)}{\partial x}=0, \quad \frac{\partial \Phi(x,y,C)}{\partial y}=0,$$

and as well as for the irregular points $C = C(x, y)$, then:

$$\frac{\partial C}{\partial x}+\frac{\partial C}{\partial y}\frac{dy}{dx}\neq 0.$$

Irregular points

Then the equation for the set of the irregular points looks as:

$$\frac{\partial \Phi(x, y, C)}{\partial C} = 0.$$

The set of irregular points

coincides with the equation for an envelope functions.
Therefore additional studies need to define kind of the points

An example

Let's consider

$$(y - C)^2 - \frac{2}{3}(x - C)^3 = 0.$$

Then the equation for the irregular points:

$$-2(y - C) + 2(x - C)^2 = 0, \quad (y - C)^2 - \frac{2}{3}(x - C)^3 = 0.$$

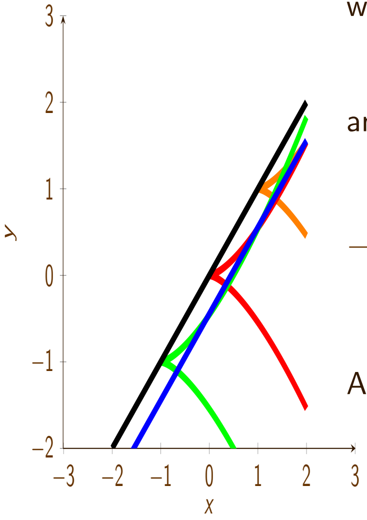
Then:

$$(y - C) = (x - C)^2, \quad (x - C)^4 - \frac{2}{3}(x - C)^3 = 0.$$

So

$$(x - C)^3(x - C - \frac{2}{3}) = 0.$$

An example



Therefore
we obtain two different curves:

$$x = C, \quad y = C, \Rightarrow y = x$$

and

$$x = C + \frac{2}{3},$$

$$-2(y - C) + 2(x - C)^2 = 0, \Rightarrow$$

$$-2\left(y - x + \frac{2}{3}\right) + \frac{4}{9} = 0$$

As a result:

$$y = x - \frac{4}{9}.$$

Clairaut's equation

Let's consider

$$y = xy' + \Psi(y').$$

To solve this equation let's differentiate it:

$$y' = y' + xy'' + y''\Psi'(y').$$

Define $y' = p$. It yields:

$$p'(x + \Psi(p)) = 0.$$

As a result one obtain two solutions:

$$p = C, \quad x + \Psi'(p) = 0.$$

Clairaut's equation

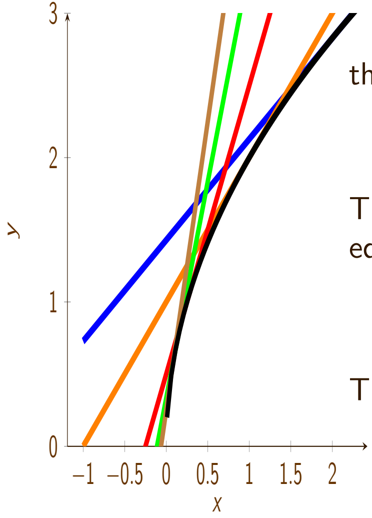
First equation defines a family of straight lines:

$$y = Cx + \Psi(C).$$

Second equation defines the special solution:

$$y = xp(x) + \Psi(p(x)), \quad p(x) : x = \Psi(p(x)).$$

An example of the Clairaut's equation



$$y = xy' + \frac{1}{y'}.$$

then the family of the straight lines:

$$y = Cx + \frac{1}{C}.$$

The

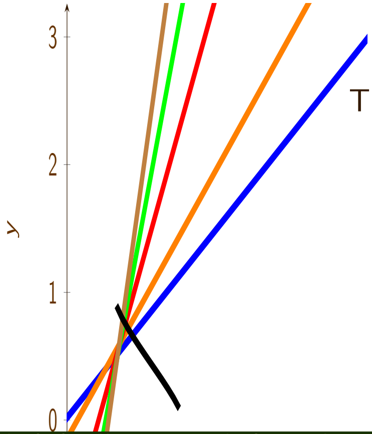
equation for an envelope function:

$$x - \frac{1}{p^2} = 0, \quad p = \pm\frac{1}{\sqrt{x}}.$$

Then:

$$y = \pm 2\sqrt{x}, \Rightarrow y^2 = 4x.$$

Cycloid



$$y = xy' + \frac{ay'}{\sqrt{1+(y')^2}}$$

The family of the curves:

$$y = Cx + \frac{aC}{1+C^2}.$$

Cycloid

The equation for the envelope function:

$$y = xp(x) + \frac{ap(x)}{\sqrt{1+p^2(x)}}, \quad x = \frac{-a}{\sqrt{1+p^2(x)}},$$

or

$$y = \frac{aC^3}{(1+C^2)^{3/2}}, \quad x = \frac{-a}{(\sqrt{1+C^2})^3},$$

excluding C one get:

$$y^{2/3} + x^{2/3} = a^{2/3}.$$

This curve is called *cycloid*.

Orthogonal trajectories

Let's consider a family of the curves:

$$\Phi(x, y, C) = 0.$$

Definition

Lines are passed through given family of curves under right angle are called *orthogonal trajectories*.

Let the family be integral curves for:

$$F(x, y, y') = 0.$$

The the coefficient of tangent line for the curve at point $M(x, y)$ is defined by a derivative y'. The coefficient of an orthogonal line looks as $-1/y'$.

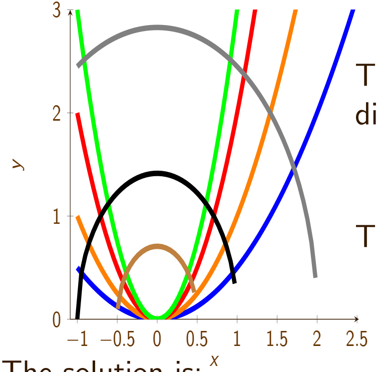
Orthogonal trajectories

Then the family of orthogonal curves is defined as:

$$F(x, y, -1/y') = 0.$$

For example in physics such trajectories define an equipotential lines of magnet and electric fields.

An example



Consider:

$$y = Cx^2.$$

The
differential equation for this family:

$$y' = 2Cx, \Rightarrow y' - 2\frac{y}{x} = 0.$$

The orthogonal trajectories:

$$\frac{1}{y'} + 2\frac{y}{x} = 0.$$

The solution is:

$$ydy = -\frac{1}{2}xdx, \quad x^2 + \frac{y^2}{2} = C^2.$$

Therefore the ellipses are orthogonal to the parabolas.

Summary

- A parabola safety
- An envelope
- Irregular points
- Clairaut's equation
- Orthogonal trajectories