Exact equations.

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Theorem about existence and uniqueness solution

Theorem

Let $f(x, y)$ be continuous with respect to x, y and be such that:

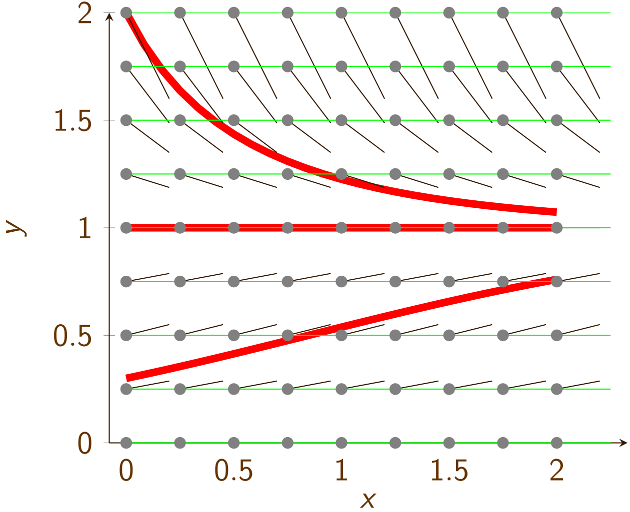
$$|f(x,y)| < b, \quad U : X_l < x < X_r, \quad Y_l < y < Y_u.$$
$$|f(x,y) - f(x,z)| < C|y - z|, (x,y) \in U,$$

then there exists unique solution of the following initial valued problem

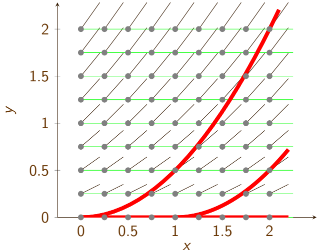
$$\frac{dy}{dx} = f(x,y), \quad y|_{x=x_0} = y_0$$

in some interval $x \in (x_0, c) \subset (X_l, X_r)$.

Trajectories of $\frac{dy}{dx} = y(1-y)$ do not intersect!



Counterexample



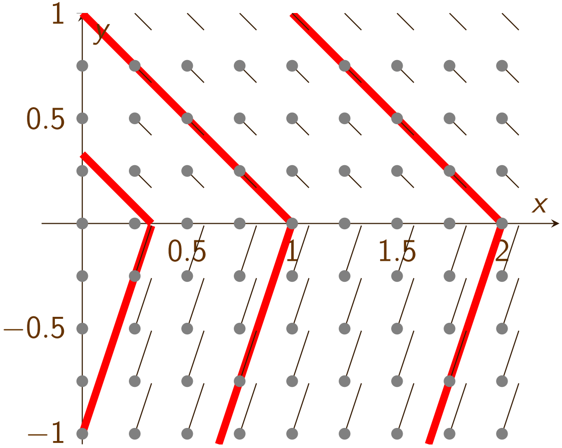
$$\frac{dy}{dx} = \sqrt{y}, \quad 2\sqrt{y} = x + c,$$

$$y = \frac{1}{4}(x + c)^2.$$

The general solution at $x = -c$ tangents to a special solution $y \equiv 0$. So the right hand side of the equation, which is $\sqrt{y}$, cannot be represented as smooth function at $y = 0$.

$$|\sqrt{y} - \sqrt{z}| < \frac{C}{2\sqrt{y}}|y - z|.$$

A direction field for $\frac{dy}{dx} = 1 - 2\text{sign}(y)$.



Generalization of the theorem

Lets consider a n-th order differential equation:

$$\frac{d^n y}{dx^n} = f(x, y, y', y'', \ldots, y^{(n-1)}), \quad \text{where} \quad y^{(k)} = \frac{d^k y}{dx^k}.$$

Define $\vec{Y} = (Y_1, \ldots, Y_n) \equiv (y, y', \ldots, y^{(n-1)})$, then the equation can be presented as a system of n first-order equations:

$$\frac{dY_1}{dx} = Y_2, \ldots, \frac{dY_{n-1}}{dx} = Y_n, \ \frac{dY_n}{dx} = f(x, \vec{Y}).$$

Examples

Let's consider the second order equations:

$$y'' + y = f(x), \quad Y_1 = y, Y_2 = y', \Rightarrow \begin{cases} Y_1' = Y_2, \\ Y_2' = -Y_1 + f(x). \end{cases}$$

$$\phi'' + \sin(\phi) = f(x), \quad Y_1 = \phi, Y_2 = \phi', \Rightarrow \begin{cases} Y_1' = Y_2, \\ Y_2' = -\sin(Y_1) + f(x). \end{cases}$$

A general case:

$$\begin{cases} Y_1' = f_1(x, Y_1, \ldots, Y_n), \\ \ldots, \\ Y_n' = f_n(x, Y_1, \ldots, Y_n). \end{cases}$$

Example

Let's define the norm of the functional vector space:

$$||\vec{Y}|| = \max_{k\in\{1,\dots,n\}} \sup_{x\in(X_l,X_r)} |y_k(x)|.$$

Let's find, for example, a value of this norm for a certain two-dimensional vector-function, say: $\vec{V} = (V_1(t), V_2(t))$, where $V_1(x) = 1 + x^2$, $V_2 = x\sin(x)$ and $x \in (0, \pi/2)$:

$$||\vec{V}|| = \max_{k\in\{1,2\}} \sup_{x\in(0,\pi/2)} \{1 + x^2, x\sin(x)\}$$

$$= \max\{1 + \frac{\pi^2}{4}, \frac{\pi}{2}\sin(\pi/2)\} = 1 + \frac{\pi^2}{4}.$$

Theorem for the system of first-order equations

Let $\vec{f}(x, \vec{y})$ be continuous with respect to $x, y_1, \ldots, y_n$ and be such that:

$$|f(x, \vec{y})| < b, \text{ for } U : X_l < x < X_r, A_k < y_k < B_k, \ k \in \{1, \ldots, n\}$$
$$||\vec{f}(x, \vec{y}) - f(x, \vec{z})|| < C||\vec{y} - \vec{z}||, (x, \vec{y}) \text{ and } (x, \vec{z}) \in U,$$

then there exists unique solution of the following initial valued problem

$$\frac{d\vec{y}}{dx} = \vec{f}(x, \vec{y}), \quad \vec{y}|_{x=x_0} = \vec{y}_0$$

in some interval $x \in (x_0, c) \subset (X_l, X_r)$.

The proof of the theorem is literally the same as for the one-dimensional first-order equation, which was considered on the previous lecture.

Formulas with a constant value of a parameter

In physics the equations in the specific form like:

$$\frac{m\dot{y}^2}{2} + k\frac{y^2}{2} = E$$

This function define the full mechanic energy of a load mass m on a spring, where k is a spring constant. The first term defines a kinetic energy and the second one – a potential energy of the spring.

Another formula of the same type for the full energy of a pendulum:

$$\frac{ml^2\dot{\phi}^2}{2} - lgm\cos(\phi) = E$$

Here m is a mass of the pendulum and l is its length, g is the gravitational acceleration.

The general form for these cases is

Initial data as a parameter of a solution

According to the theorem about existence of a unique solution for a given initial valued problem, the solution can be written as an function of one external parameter. In general case the solution can be written as follows:

$$y = y(x, C), \quad \text{where} \quad C \equiv C(y_0, x_0).$$

This parameter defines the integral curve as a trajectory on the vector field for the given first-order differential equation.

Exact equations

Let's try rewrite the function $y = y(x, C)$ in the implicit form: $F(x, y) = C$. The differential of the function $F(x, y)$:

$$dF(x, y) \equiv \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial x} dx = 0.$$

Definition: exact equation

An equation

$$\frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial x} dx = 0.$$

which can be represented as

$$dF(x, y) = 0$$

is called **exact equation.**

Properties of the exact solution

Theorem

If coefficients of the equation

$$A(x, y)dy + B(x, y)dx = 0$$

are such that their partial derivative exist and

$$\frac{\partial A(x, y)}{\partial x} \equiv \frac{\partial B(x, y)}{\partial y},$$

then the differential equation is the exact equation.

Properties of the exact solution

Another words we can find the function $F(x, y)$ such that:

$$A(x, y) = \frac{\partial F(x, y)}{\partial y}, \quad B(x, y) = \frac{\partial F(x, y)}{\partial x},$$

and the equation can be considered as the differential of the function $F(x, y)$:

$$dF(x, y) = 0.$$

Proof of the theorem about an exact equation

- Suppose that there exists such double differentiable function F, then $dF(x, y) = 0$ can be rewritten as follows

$$dF(x, y) \equiv \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial x} dx = 0.$$

Define $A(x, y) = \frac{\partial F}{\partial y}$, $B(x, y) = \frac{\partial F}{\partial x}$, then we obtain:

$$A(x, y)dy + B(x, y)dx = 0$$

and due to existence of the double derivatives of $F(x, y)$

$$\frac{\partial A}{\partial x} \equiv \frac{\partial^2 F}{\partial x \partial y}, \quad \frac{\partial B}{\partial y} \equiv \frac{\partial^2 F}{\partial y \partial x}, \quad \frac{\partial^2 F}{\partial x \partial y} \equiv \frac{\partial^2 F}{\partial y \partial x}$$

Proof of the theorem about an exact equation

- Let's suppose

$$A = \frac{\partial F}{\partial y}, \quad B = \frac{\partial F}{\partial x}.$$

Then

$$F = \int_{y_0}^{y} A(x,y)dy + f_1(x), \quad F = \int_{x_0}^{x} B(x,y)dx + f_2(y).$$

Using the condition of the theorem we get

$$F(x,y) \equiv \int_{y_0}^{y} A(x,y)dy + f_1(x) + \mathrm{const},$$

or the same case:

$$F(x,y) \equiv \int_{x_0}^{x} B(x,y)dx + f_2(y) + \mathrm{const},$$

Examples

- The equation for the circles, defined by the formula $y^2 + x^2 = R^2$ is following:

$$2ydy + 2xdx = 0, \Rightarrow ydy + xdx = 0.$$

- The equation for the family of hyperbolas: $y^2 - x^2 = \mathrm{const}$. The differential equation for these curves is:

$$2ydy - 2xdx = 0, \Rightarrow ydy - xdx = 0.$$

Theorem about existence of integrating factor

The corollary of the theorem of existence of unique solution for initial valued problem can be formulated as follows.

Theorem about existence of integrating factor

For any equation in the differential form

$$A(x, y)dx + B(x, y)dy = 0$$

there exists an multiplier $\mu(x, y)$ such that the equation

$$\mu(x, y)A(x, y)dx + \mu(x, y)B(x, y)dy = 0$$

is the exact equation.

Example of usage the integrating factor

The equation

$$x^2y^2dy + (xy^3 - 1)dx = 0$$

is not exact equation because

$$\frac{d}{dx}(x^2y^2) \neq \frac{d}{dy}(xy^3 - 1).$$

The integrating factor is $\mu(x, y) \equiv x$:

$$\frac{d}{dx}(x^3y^2) = 3x^2y^2, \quad \frac{d}{dy}(x^2y^3 - 1) = 3x^2y^2.$$

Then the equation

$$x(x^2y^2dy + (xy^3 - 1)dx) = 0$$

is the exact equation.

Notice

Deriving an integrating factor for a general equation is a challenging problem.

Linear homogeneous first-order equations

A homogeneous first-order linear differential equation is a differential equation in the form:

$$y' + P(x)y = 0.$$

where $P(x)$ is a known function. The term "linear"means that the equation can be expressed in terms of first-order derivatives only, and does not involve any products or powers of the unknown function y:
Let's $\lambda = \mathrm{const}$ and define new function $\lambda y(x) = \tilde{y}(x)$, then the substitution this function into the equation instead of $y(x)$ produces the same equation for $y(x)$:

$$\tilde{y}' + P(x)\tilde{y} = 0 \Rightarrow \lambda y' + P(x)\lambda y = 0, \Rightarrow \lambda(y' + P(x)y) = 0.$$

Examples of non-homogeneous equations

The differential equation in the form:

$$y' = y^2, \quad \lambda y \to \tilde{y}, \Rightarrow \tilde{y}' = \tilde{y}, \Rightarrow$$
$$\lambda y' = \lambda^2 y^2, \Rightarrow y' = \lambda y^2.$$

Yet an another example:

$$y' = \sin(y), \quad \lambda y \to \tilde{y}, \Rightarrow \tilde{y}' = \sin(\tilde{y}), \Rightarrow$$
$$\lambda y' = \sin(\lambda y), \Rightarrow y' = \frac{1}{\lambda}\sin(\lambda y).$$

Integration of homogeneous linear equations

Consider the first-order linear differential equation:

$$y'(x) + f(x)y(x) = 0$$

we can solve for $y(x)$ using the method of separation of variables.
First, we multiply both sides of the equation by dx and divide both sides by y to get:

$$\frac{dy(x)}{y} + f(x)dx = 0.$$

Integrating both sides with respect to x, we obtain:

$$\log|y| - \log|C| = -\int f(x)dx,$$

which implies that:

$$y = Ce^{-\int f(x)dx}$$

Integration of non-homogeneous linear equations

To solve non-homogeneous first-order linear differential equations of the form:

$$y'(x) + f(x)y(x) = Q(x)$$

where $P(x)$ and $Q(x)$ are known functions. To solve this equation, we first find the general solution $u(x)$ to the corresponding homogeneous equation:

$$u'(x) + f(x)u(x) = 0.$$

Suppose one knows a certain solution of the non-homogeneous equation $h(x)$:

$$h' + f(x)h = Q(x),$$

then the general solution of the non-homogeneous equation is:

$$y(x) = Cu(x) + h(x), \quad \forall C \in \mathbb{R}.$$

Method of variable of parameter.

To find a certain solution of the non-homogeneous equation let's try to consider $h(x) = C(x)u(x)$, where $u(x)$ is a solution of the complement homogeneous equation and $C(x)$ is new unknown function.
Substitute the form $h(x)$ into the non-homogeneous equation. It yields:

$$C'u + Cu' + f(x)Cu = Q, \Rightarrow C'u + C(u' + f(x)u) = Q, \Rightarrow C'u = Q(x).$$

Then

$$C' = \frac{Q(x)}{u(x)}, \Rightarrow dC = \frac{Q(x)}{u(x)}dx, \Rightarrow C = \int_{x_0}^{x} \frac{Q(t)}{u(t)}dt.$$

Then a particular solution is: $h(x) = u(x) \int_{x_0}^{x} \frac{Q(t)}{u(t)}dt.$

Theorem about solution of a non-homogeneous linear first-order equation

Theorem

A general solution of the first-order non-homogeneous equation for $y' + f(x) = Q(x)$ can be written in the form:

$$y(x) = Cu(x) + u(x)\int_{x_0}^{x} \frac{Q(t)}{u(t)} dt,$$

where $u(x)$ is a solution of the complementary homogeneous equation $u' + f(x)u = 0$ and x_0 is some constant.

To **proof** this theorem one should differentiate the function $y(x)$ with respect to x.

An example

$$y' + y = x^2 + x.$$

The solution of the complementary linear homogeneous equation:

$$v' + v = 0, \quad v = Ce^{-x}.$$

A particular solution of the given non-homogeneous equation is follows:

$$\begin{aligned} h(x) &= e^{-x}\int_0^x (t^2+t)e^t dt = e^{-x}((x^2-x+1)e^x - 1) \\ &= (x^2-x+1) - e^{-x}. \end{aligned}$$

Then the general solution can be written as follows:

$$y = Ce^{-x} + (x^2 - x + 1).$$

The Bernoulli equations

Equations in the following form

$$y' + f(x)y = Q(x)y^n, \quad n \neq 1$$

are called as Bernoulli equations in the honor of Jacob Bernoulli.

These equations can be rewritten as linear ones after dividing both part by y^n

$$\frac{y'}{y^n} + \frac{f(x)}{y^{n-1}} = Q(x)$$

and changing $v = y^{1-n}, \;\; v' = (1-n)\frac{y'}{y^n}$. As a result one gets a non-homogeneous linear equation of the first-order for $v(x)$:

$$\frac{v'}{(1-n)} + f(x)v = Q(x), \Rightarrow v' + (1-n)f(x)v = (1-n)Q(x).$$

An example

$$y' + xy = y^3.$$

Divide both parts by y^3,then we get:

$$\frac{y'}{y^3} + \frac{x}{y^2} = 1.$$

Define $v = y^{-2}$, then $v' = -2y^{-3}$ and

$$\frac{v'}{-2} + xv = 1, \Rightarrow v' - 2xv = -2.$$

As a result we have derived the non-homogeneous linear first-order equation, which solution can be constructed by the approach discussed above.

Summary

Theorem about existence of unique solution and corollaries

Exact equations

Linear first-order equations

Bernoulli equations