The existence and uniqueness of solution of first order DE

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Special form of first order equations

Theorem of existence of solution

Integrating of the separable equations

Explicit and implicit forms of equations

We will say that a first order equation is written in *explicit form* if the first derivative looks as an explicit function of dependent, say y, and independent, say x, variables:

$$\frac{dy}{dx} = f(x, y).$$

The same equation can be represented as

$$\frac{dx}{dy} = \phi(x, y), \quad \phi(x, y) = \frac{1}{f(x, y)}.$$

The equation is written in an *implicit form* if it looks like

$$F\left(x, y, \frac{dy}{dx}\right) = 0, \quad \text{or} \quad \Phi\left(x, y, \frac{dx}{dy}\right) = 0.$$

Examples of equations in explicit and implicit forms

Equations in explicit forms:

$$\frac{dy}{dx} = \frac{x}{y} \Leftrightarrow \frac{dx}{dy} = \frac{y}{x}$$

An equation in implicit forms:

$$\frac{1}{2}\left(\frac{dy}{dx}\right)^2 + \frac{1}{2}y^2 - 1 = 0 \Leftrightarrow \left(\frac{dx}{dy}\right)^2 + \frac{1}{y^2 - 2} = 0$$

The equation in the implicit and explicit forms

$$\frac{1}{2}\left(\frac{dy}{dx}\right)^2 + \frac{1}{2}y^2 - 1 = 0 \Leftrightarrow \frac{dy}{dx} = \left\{ \begin{array}{c} \sqrt{2 - y^2}, \\ -\sqrt{2 - y^2}. \end{array} \right.$$

A pendulum

A mechanical energy of the pendulum:

$$E = m\frac{(l\dot{\phi})^2}{2} - mgl\cos(\phi).$$

This formula can be considered as an implicit form of differential equation for the instant position of the pendulum, which is defined by the angle ϕ.

$$\dot{\phi} = \pm\sqrt{2E + \frac{g}{l}\cos(\phi)} \Leftrightarrow \frac{d\phi}{dt} = \pm\sqrt{2E + \frac{g}{l}\cos(\phi)}.$$

A differential form of the equation

A differential of function change is a linear part of the additional value of the function:

$$dy = \frac{dy}{dx}dx.$$

So the differential equation for some function y we can write as an *equation in the differentials*:

$$dy - f(x,y)dx = 0, \quad dx - \phi(x,y)dy = 0.$$

A general form of equation in differentials looks like:

$$h(x,y)dy + g(x,y)dx = 0.$$

A parametric form

Th forth form of the same equation we can derive from assumption for the parametric form of the solution:

$$y = y(t), \quad x = x(t), \quad t \in \mathbb{R}.$$

In this case

$$dx = \frac{dx}{dt}dt, \quad dy = \frac{dy}{dt}.$$

Then the equation can be represented as

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g(x,y)}{h(x,y)}.$$

So we can write the equation as a system of differential equation as a parametric definition of the integral curve:

$$\frac{dx}{dt} = h(x,y), \quad \frac{dy}{dt} = g(x,y).$$

Example: $\frac{dy}{dx} = ky$.

Let's integrate left part of the equation over x:

$$\int \frac{dy}{dx} dx \equiv \int dy.$$

The same gimmick does not work with the right hand side:

$$\int y dx \equiv \int y(x) dx$$

because the integrand contains unknown function $y(x)$.

The solution in a series form

Consider the initial value problem

$$\frac{dy}{dx} = y, \quad y_{x=0} = y_0.$$

It is easy to check the solution of the integral equation like

$$y = y_0 + \int_0^x y(\xi) d\xi$$

gives the solution of this initial valued problem.
Let's differentiate the integral equation:

$$\frac{dy}{dx} = \frac{d}{dx}\left(A + \int_0^x y(\xi) d\xi\right) \equiv y(x).$$

An equivalence of initial value problem for differential and integral equation

As a result we obtain the statement:

The solution of the integral equation

$$y = y_0 + \int_0^x y(\xi) d\xi$$

coincides to the initial value problem for the differential equation:

$$\frac{dy}{dx} = y, \quad y|_{x=0} = y_0.$$

Constructing the series

Define a recurrent sequence:

$$y_{n+1}(x) = y_0 + \int_0^x y_n(\xi) d\xi$$

A a result we obtain:

$$y_1 = y_0 + y_0 x,$$

$$y_2 = y_0 + y_0 x + y_0 \frac{x^2}{2},$$

$$y_3 = y_0 + y_0 x + y_0 \frac{x^2}{2} + y_0 \frac{x^3}{3!},$$

$$y_{n+1} = y_n(x) + y_0 \frac{x^{n+1}}{(n+1)!},$$

$$y(x) = y_0 \left(1 + x + \frac{x^2}{2} + \dots + \frac{x^n}{n!} + \dots \right) \Leftrightarrow y(x) = y_0\, e^x.$$

A general approach

Consider an initial value problem:

$$\frac{dy}{dx} = f(x, y), \quad y|_{x=x_0} = y_0.$$

Connect an integral equation with the problem:

$$y(x) = y_0 + \int_{x_0}^{x} f(\xi, y(\xi))d\xi.$$

It easy to check by straightforward differentiation that the integral equation and initial value problem define the same function if such function exists.

A recurrent process. Integral operator.

Define a recurrent sequence on some interval $x \in (0, a)$:

$$y_{n+1} = y_0 + \int_{x_0}^{x} f(\xi, y_n(\xi)) d\xi.$$

On this step we need to assume that the integrand is bounded in area on plane (x, y), for simplicity in some rectangle:

$$|f(x, y)| < b, \quad X_l < x < X_r, \quad Y_l < y < Y_u.$$

Therefore y_k remains in the interval $Y_l < y < Y_u$ as $Y_l - y_0 < b|x - x_0| < Y_u - y_0$. This inequalities defines the interval of x which can be used for the recurrent sequence.

A functional space and metrics

We will say that two function are equivalent on $x \in (a, b)$ if

$$\sup_{x \in (a,b)} |y(x) - z(x)| = 0.$$

The *functional* which connect a function $y(x)$ and a number in $\mathbb{R}$:

$$L(y(x)) := ||y(x)||$$

such that

- $||y(x)|| \geq 0.$
- If $||y(x)|| = 0$, then $y(x) \equiv 0.$
- $\forall \lambda > 0, ||\lambda y(x)|| = \lambda ||y(x)||$
- $||y(x) - z(x)|| \leq ||y(x) - u(x)|| + ||u(x) - z(x))||.$

will be named a norm or metrics.

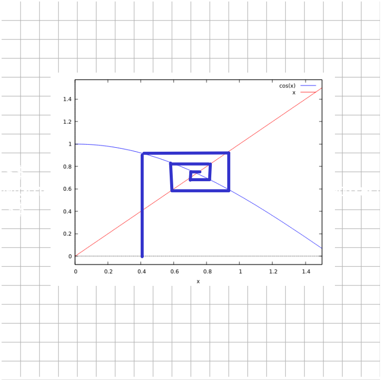
A pointwise norm

The functional

$$||y(x)|| = \sup_{x\in(a,b)} |y(x)|$$

defines a *pointwise* norm. We will consider continuous and bounded functions. The space of such functions we will mark C and $y(x) \in C$.
This approach expands concept of distance on the the functional space.

A fixed point of contracting map



Below
we will consider the formula

$$y_{n+1} = y_0 + \int_{x_0}^{x} f(\xi, y_n(\xi)) d\xi.$$

as a constructing
map in the functional space.
Analogously
to the map like follows:

$$x = \phi(x)$$

in the real numbers, but instead of the point $x \imath \mathbb{R}$ we consider the point in the functional space: $y(x) \in C$.

A recurrent process. An estimation of the difference.

Estimate a difference between y_{n+1} and y_n as follows:

$$\sup_{x\in(0,a)} |y_{n+1}(x) - y_n(x)| =$$

$$\sup_{x\in(0,a)} \left| \int_{x_0}^{x} f(\xi, y_n(\xi))d\xi - \int_{x_0}^{x} f(\xi, y_{n-1}(\xi))d\xi \right|.$$

To get the estimate for the difference we need additional constraint for the smooth of $f(x, y)$:

$$|f(x, y) - f(x, z)| \leq C|y - z|, \; C > 0,$$
$$as \; X_l < x < X_r, \quad Y_l < y < Y_u.$$

An example

$$f(x, y) \equiv y^2.$$

Then:

$$f(y) - f(z) = y^2 - z^2 = (y + z)(y - z).$$

Hence

$$|y^2 - z^2| \leq |y + z| \cdot |y - z|, \quad |y + z| < L = \text{const}, \quad y, z \in (a, b).$$

So:

$$|y^2 - z^2| \leq L \cdot |y - z|, \quad y, z \in (a, b).$$

A recurrent process. A convergent sequence.

Define G interval x such that both inequalities are true:
$C|x - x_0| < 1$, $Y_l - y_0 < b|x - x_0| < Y_u - y_0$.

$$\sup_{x \in G} |y_{n+1}(x) - y_n(x)| =$$

$$\sup_{x \in G} \left| \int_{x_0}^{x} f(\xi, y_n(\xi)) d\xi - \int_{x_0}^{x} f(\xi, y_{n-1}(\xi)) d\xi \right| \leq$$

$$C|x - x_0| \sup_{x \in G} |y_n(x) - y_{n-1}(x)| =$$

$$q \sup_{x \in G} |y_n(x) - y_{n-1}(x)|, \quad 0 < q < 1.$$

As a result we get:

$$\sup_{x \in G} |y_{n+1}(x) - y_n(x)| < q \sup_{x \in G} |y_n(x) - y_{n-1}(x)|, \quad 0 < q < 1.$$

This means the sequence is convergent.

A recurrent process. A uniqueness of the solution.

Let's suppose there exist two different solutions $Y(x)$ and $Z(x)$, then:

$$\sup_{x\in G}|Y(x)-Z(x)| < q\sup_{x\in G}|Y(x)-Z(x)|, \quad 0<q<1.$$

Hence $\sup_{x\in G}|Y(x)-Z(x)|=0$ and there is unique solution of the integral equation.

Corollary

The recurrent process converges to the unique solution of the integral equation.

Theorem about existence and uniqueness solution

Theorem

Let $f(x, y)$ be continuous with respect to x, y and be such that:

$$|f(x, y)| < b, \quad U : X_l < x < X_r, \quad Y_l < y < Y_u.$$
$$|f(x, y) - f(x, z)| < C|y - z|, (x, y) \in U,$$

then there exists unique solution in some interval $x \in (x_0, c)$.

Counterexample

$$\frac{dy}{dx} = \sqrt{y}, \quad 2\sqrt{y} = x + c, \quad y = \frac{1}{4}(x+c)^2.$$

The general solution at $x = -c$ tangents to a special solution $y \equiv 0$. So the right hand side of the equation, which is $\sqrt{y}$ is cannot be represented as smooth function as $y \sim 0$.

$$|\sqrt{y} - \sqrt{z}| < \frac{C}{2\sqrt{y}}|y - z|.$$

Integration by separation $\frac{dy}{dx} = ky$

The main idea of variable separation

Rewrite, if it is possible, the equation in such way that integrand be written in explicit form.

If $y = 0$ then $y \equiv 0$ is the trivial solution.

$$\frac{dy}{dx} = ky, y \neq 0 \Leftrightarrow \frac{1}{y}\frac{dy}{dx} = k.$$

$$\int \frac{dy}{y} = \int k dx,$$

It yields:

$$\log(|y|) = kx + c,$$

Define $c = \log(|C|)$, then

$$\log(|y|) = kx + \log(|C|) \Leftrightarrow y = C\, e^{kx}.$$

Integration of initial value problem

Consider the initial value problem:

$$\frac{dy}{dx} = ky, \quad y|_{x=x_0} = y_0, \quad y_0 \neq 0.$$

In this case the antiderivatives should be changed by definite integrals:

$$\int_{y_0}^{y} \frac{dy}{y} = \int_{x_0}^{x} kdx.$$

It yields:

$$\log(|y|) - \log(|y_0|) = e^{k(x-x_0)}.$$

The same formula can be represented as follows:

$$y = y_0 e^{k(x-x_0)} \Leftrightarrow y = Ae^{kx}, \quad A = y_0 e^{-k\,x_0}.$$

Separated equation in general form

Consider the equation in the form:

$$\frac{dy}{dx} = g(x)h(y).$$

All straight lines $y \equiv y_k$, for $h(y_k) = 0$ are constant solutions of the equation.

Example

$$\frac{dy}{dx} = g(x)(y+1)(y^2-3), \quad y \equiv 1, \quad y \equiv \sqrt{3}, \quad y \equiv -\sqrt{3}.$$

Separated equation in general form

- Consider interval of bounded and nonzero values of the right-hand side for the equation

$$\frac{dy}{dx} = g(x)h(y).$$

- Rewrite the equation in the form: $\frac{1}{h(y)}\frac{dy}{dx} = g(x)$.
- Integrate both part over x: $\int \frac{dy}{h(y)} = \int g(x)dx$.
- In case of initial valued problem $y|_{x=x_0} = y_0$ the answer should be presented in the form

$$\int_{y_0}^{y} \frac{dy}{h(y)} = \int_{x_0}^{x} g(x)dx.$$

The two last forms can be considered as the solution in quadrature.

An example. Equation for the circle

Let's consider the equation

$$\frac{dy}{dx} = -\frac{x}{y}.$$

This form of the equation assumes that on the axis $y = 0$ this equation couldn't be considered. However we might rewrite this equation in the following form:

$$\frac{dx}{dy} = -\frac{y}{x}.$$

This form of the equation highlights that there do not solutions as $x = 0$.

But in reality these restrictions only explain that the derivatives in the left-hind sides might be infinite at $x = 0$ or $y = 0$. However while $x \neq 0$ nor $y \neq 0$ the solutions of these equations coincides.

Equation for the circle

Let's consider an equation in differential form:

$$xdx + ydy = 0, \quad \text{or} \quad xdx = -ydy$$

integrate both parts:

$$\int xdx = -\int ydy, \quad \frac{x^2}{2} = -\frac{y^2}{2} + C$$

As a result we obtain:

$$\frac{x^2}{2} + \frac{y^2}{2} = C.$$

Here $C > 0$ and the integral curve is a circle with radius $r = \sqrt{C}$ and $2C = x_0^2 + y_0^2$ for given initial values of x and y.

Example: a logistic equation $\frac{dy}{dx} = y(1-y)$

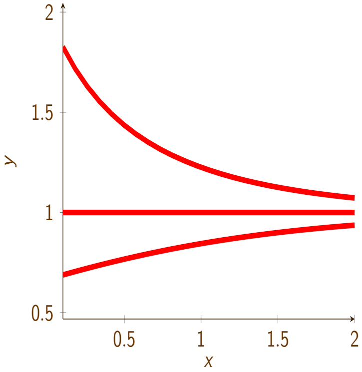
- Two constant solutions $y \equiv 0$ and $y \equiv 1$.
- A general solution in quadrature form:

$$\int \frac{dy}{y(1-y)} = \int dx$$

Integral in the left-hand side can be represented as follows:

$$\int \frac{dy}{y(1-y)} = \int \frac{dy}{y} + \int \frac{dy}{1-y}$$
$$= \log(|y|) - \log(|1-y|) = \log\left(\left|\frac{y}{1-y}\right|\right)$$

Solution of the logistic equation



So,

$$\frac{y}{1-y} = ce^x, \quad y(x) = \frac{ce^x}{1+ce^x}.$$

In more convenient form a general solution of the logistic equation has the form:

$$y(x) = \frac{1}{1+Ce^{-x}},$$

where $C = 1/c$.

Summary

- A variety of formulae for the first order differential equations.
- A norm in a functional space.
- Theorem about existence and uniquince of solution of the first order differential equation.
- Separated equations.
- An example. A general solution of the logistic equation.