Short introduction into Differential equations

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I.Newton and applications of mathematics

Differential equations were intorduced into mechanics by Isaac Newton (1642-1727) in his the most famous work "Philosophiæ Naturalis Principia Mathematica"published in 1687.

- Let's define t as an instant value of time.
- Any strightforward motion of matherial point will be defined as a function $x = x(t)$, where x is a distance between an origin and instant position of the point.
- Following by Newton **first derivative** of x, which is $\dot{x} = v(t)$ is a velocity of given matherial point.
- **Second derivative** $\ddot{x}$ or first derivative of the velocity $\dot{v} = a(t)$ is an **instant acceleration** of the matherial point.

The mass in a constant gravity field

An example of differential equation

Consider a material point with constant mass $m[kg]$ and a knowing constant force like gravity of the Earth $P = mg$, where $g \sim 9.81 \left[\frac{m}{sec^2}\right]$ which acts on this point.

$$m\dot{v} = P, \quad v = \frac{P}{m}t + \text{const.}$$

An initial velocity of the point is equal to v_0:

$$v\big|_{t=t_0} = v_0, \quad v(t) = \frac{P}{m}(t - t_0) + v_0.$$

The loan interest

Suppose one person is owner of n_0 quantities of money. The that person rates these mone for someone under special condition.

Every moment of time the debt grows proportional the quantity of the debt:

$$\frac{dn}{dt} = kn, \quad n\big|_{t=t_0} = n_0.$$

$$n = n_0 e^{k(t-t_0)}.$$

A logistic equation

Let us consider a pond with carps.

- The instant number of carps is $N[carp]$.
- Growth of the population is proportional of the numbers of the fish and quantity of foods with a coefficient $k\left[\frac{1}{carp\cdot sec}\right]$
- The stable number of the carps is $M[carp]$.

These assumptions yield the equation:

$$\frac{dN}{dt} = k(M-N)N.$$

Dimensionless form of equations

Primary principle

- Mathematical formulae gain an universality when they lose certainty!

A mass in a gravitation field $m\dot{v} = P$

- A typical time scale is T seconds, define $\tau = \frac{t[sec]}{T[sec]}$ and τ is dimensionless variable.
- Let's define a typical scale of the velocity as $V\left[\frac{m}{sec}\right]$ and derive a dimensionless function $\nu = \frac{v[m/sec]}{V[m/sec]}$.
- Dimension for the force: $P\left[\frac{kg \cdot m}{sec \cdot sec}\right]$.

The equation in dimensionless form

$$m\frac{d(V\nu)}{d(T\tau)} = P \Leftrightarrow \frac{V}{T}\frac{d\nu}{d\tau} = \frac{P}{m} \Leftrightarrow \frac{d\nu}{d\tau} = \frac{P}{m}\frac{T}{V}.$$

Define $\kappa = \frac{PT}{mV}$. Dimension of κ: $\frac{[kg]\cdot[m]\cdot[sec]}{[sec]\cdot[sec]} \cdot \frac{[sec]}{[kg][m]} = \frac{1}{1}$.

Equation for the linear function

$$\frac{d\nu}{d\tau} = \kappa$$

The equation defines a linear function

$$\nu(\tau) = \kappa\tau + \text{const.}$$

Therefore this dimensionless form is an universal differential model for a lot of processes:

- the velocity of a free mass under a constant gravitational field,
- the velocity of a free electric chagre under a constant electric field,
- a salary with the stable earn.

The loan interest

$$\frac{dn}{dt} = kn.$$

- a typical time scale is T seconds, define $\tau = \frac{t[sec]}{T[sec]}$ and τ is dimensionless variable;
- a typical value is N rubles, define dimensionless function $\nu = \frac{n[rubles]}{N[rubles]}$;
- a dimension of $k \frac{1}{[sec]}$

$$\frac{d(N\nu)}{d(T\tau)} = k(n\nu), \quad \frac{N}{T}\frac{d\nu}{d\tau} = kN\nu, \quad \frac{d\nu}{d\tau} = (kT)\nu.$$

Define $\lambda = kT$. Dimension of λ is:

$$[\lambda] = \frac{[sec]}{[sec]} = \frac{1}{1}.$$

The equation for an exponent function

$$\frac{d\nu}{d\tau} = \lambda\nu.$$

This equation defines an exponential function:

$$\nu = Ce^{\lambda\tau}, \quad C = \text{const}.$$

If $\lambda > 0$ then the solution increases.
If $\lambda < 0$ then the solution decreases.
There are a lot of examples for such mathematical model:

- loan interest,
- population growth,
- nuclear decay,
- cooling down,
- penetration of new words into a language.

Dimensionless form of $\frac{dN}{dt} = k(M-N)N$

Let us derive dimensionless equation for the number of the fish. Define $\nu = N/M, \quad \kappa\left[\frac{1}{sec}\right] = kM$. It yields:

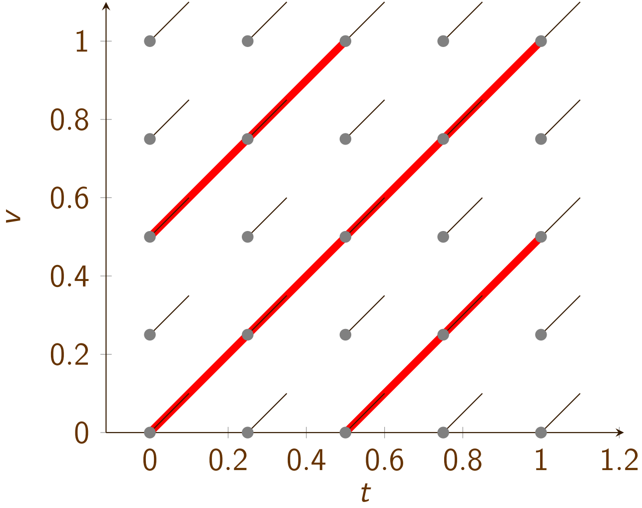
$$M\frac{d\nu}{dt} = \frac{\kappa}{M}\cdot(M - M\nu)M\nu$$

Let's rewrite the equation using a new dimensionless variable $\tau = t\kappa$. As a result we obtain

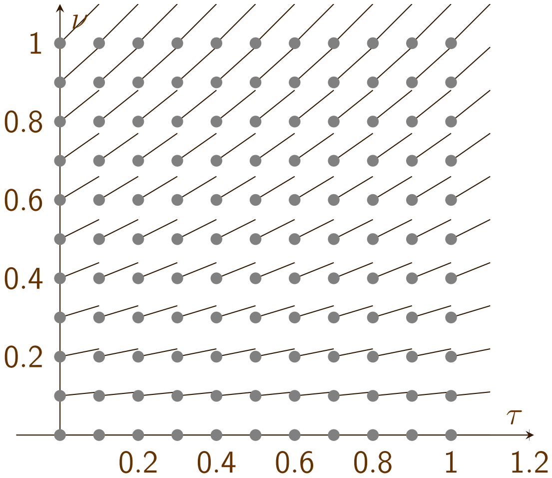
a logistic equation

$$\frac{d\nu}{d\tau} = (1-\nu)\nu.$$

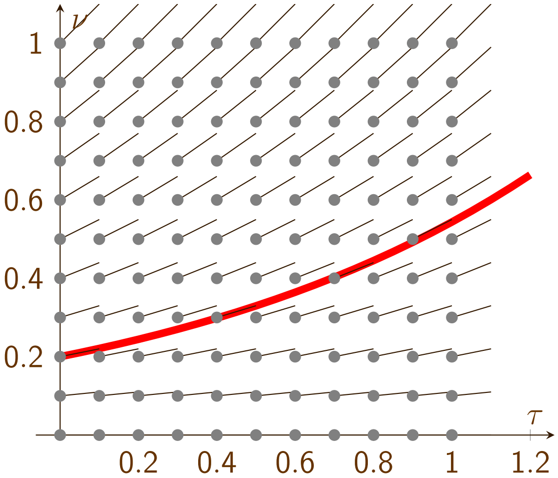
A direction field for the free body under gravitational field $\frac{dv}{dt} = 1$



A direction field for the equation $\frac{d\nu}{d\tau} = \nu$.



An integral curve $0.2\exp(\tau)$



Isocline method for solving differential equation

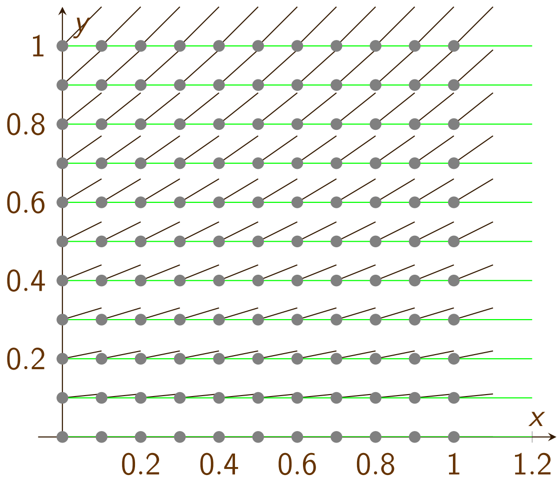
Isoclines

The curves on the plane (x, y) whether $f(x, y) = \text{const}$ is called by *isoclines*.

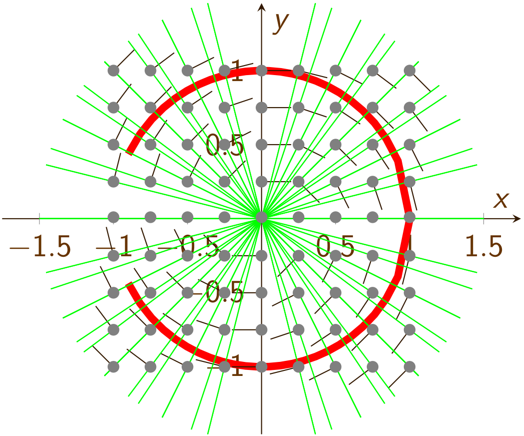
Using the isoclines one can draw the integral line as a solution of the differential equation:

$$\frac{dy}{dx} = f(x, y).$$

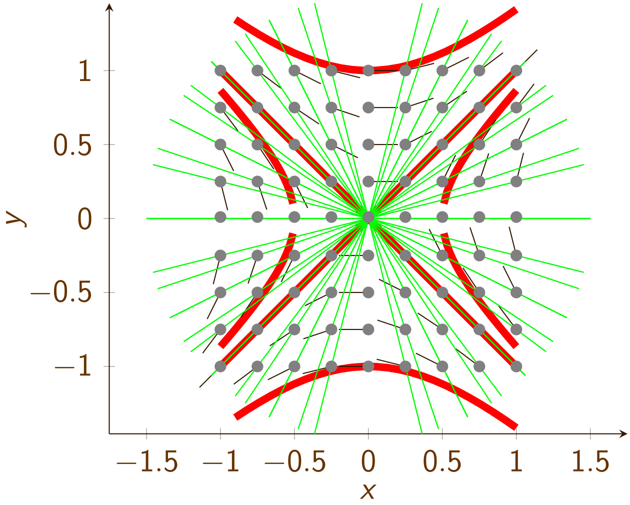
Isoclines and a direction field for $\frac{dy}{dx} = y$



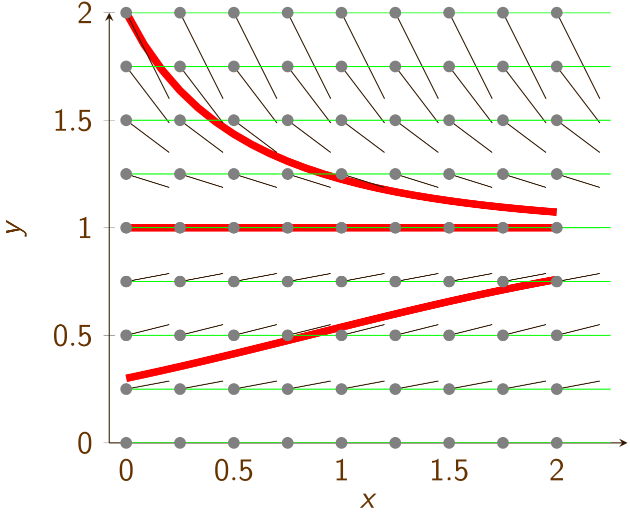
Isoclines and a direction field for $\frac{dy}{dx} = -\frac{x}{y}$.



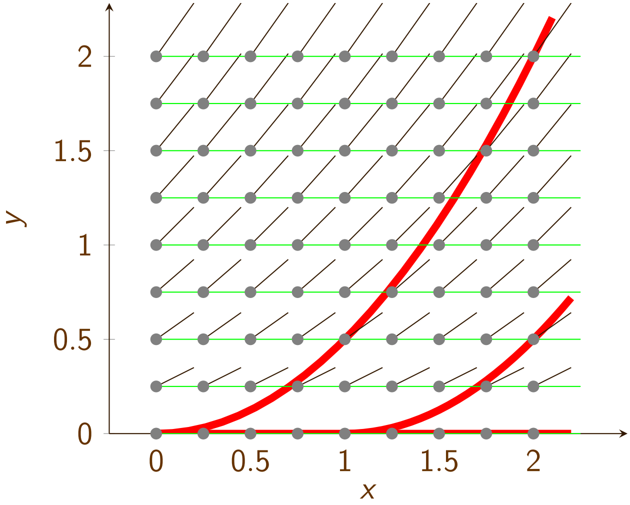
Isoclines and a direction field for $\frac{dy}{dx} = \frac{x}{y}$.



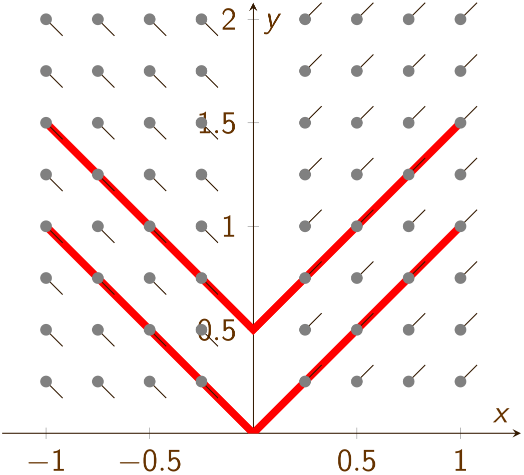
Isoclines and a direction field for $\frac{dy}{dx} = y(1 - y)$.



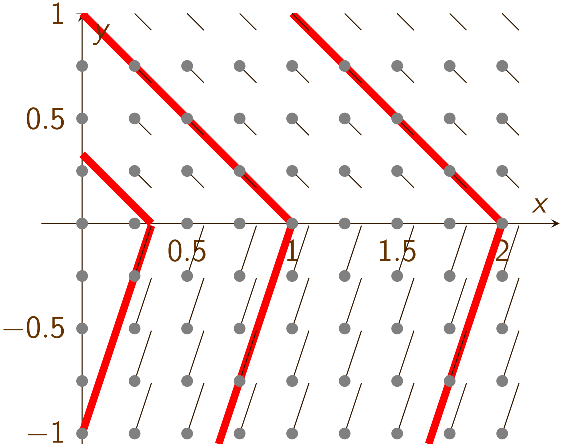
Isoclines and a direction field for $\frac{dy}{dx} = \sqrt{y}$.



Isoclines and a direction field for $\frac{dy}{dx} = \operatorname{sign}(x)$.



Isoclines and a direction field for $\frac{dy}{dx} = 1 - 2\mathrm{sign}(y)$.



Steps for graphical solution of DE

Let's construct graphical solution of the equation in the form:

$$\frac{dy}{dx} = f(x, y).$$

- Define the domain of the right-hand side function.
- Define a family of isoclinic curve like as equation $f(x, y) = r$ for a lot of values of parameter k.
- Draw the direction field on the plane (x, y).
- Starting from a given point of the plane draw the integral curve as a tangent with respect to direction field.

Disadvantages of the graphical method for solution

Disadvantages

- The graphical solution approximate the true solution.
- We cannot define global solution due to loss of accuracy.
- We cannot define the properties of the solution in neighborhood of irregular points.

Summary

- To define an uqnique solution of the DE of the first order one need an intial value of the independent variable and the value of the solution in this point.
- A graphical solution can show qualitative bechaviour of the solution.
- In special cases the trajectories of the forst order solution can be intersected.