Sapienti sat

O.M. Kiselev
o.kiselev@innopolis.ru

Innopolis university

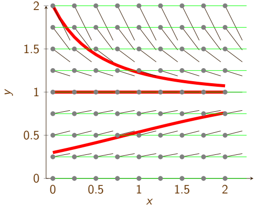
Outlines

I.Newton and applications of mathematics

Differential equations were introduced into mechanics by Isaac Newton (1642-1727) in his the most famous work "Philosophiæ Naturalis Principia Mathematica"published in 1687.

- Let's define t as an instant value of time.
- Any straightforward motion of material point will be defined as a function $x = x(t)$, where x is a distance between an origin and instant position of the point.
- Following by Newton **first derivative** of x, which is $\dot{x} = v(t)$ is a velocity of given material point.
- **Second derivative** $\ddot{x}$ or first derivative of the velocity $\dot{v} = a(t)$ is an **instant acceleration** of the material point.

A first-order differential equation defined a direction field on a plane



Steps for constructing a graphical solution for the equation $\frac{dy}{dx} = f(x, y)$.
1. Define the
domain of the right-hand side function.
2. Define a family of isoclinic
curve like as equation $f(x, y) = r$
for a lot of values of parameter k.
3. Draw the direction field on the plane
(x, y).
4. Starting from a given point of the plane draw the integral curve as a tangent with respect to direction field.

Theorem about existence and uniqueness solution

Let's consider an initial value problem:

$$\frac{dy}{dx} = f(x, y), \; y|_{x=x_0} = y_0.$$

Theorem

Let $f(x, y)$ be continuous with respect to x, y and be such that:

$$\begin{aligned} \exists U : X_l < x < X_r, \quad Y_l < y < Y_u, \; (x_0, y_0) \in U; \\ |f(x, y)| < b, \quad (x, y) \in U; \\ |f(x, y) - f(x, z)| < C|y - z|, (x, y) \in U, \end{aligned}$$

then there exists unique solution in some interval $x \in (x_0, c)$.

Integration of non-homogeneous linear equations

To solve non-homogeneous first-order linear differential equations of the form:

$$y'(x) + f(x)y(x) = Q(x)$$

where $P(x)$ and $Q(x)$ are known functions. To solve this equation, we first find the general solution $u(x)$ to the corresponding homogeneous equation:

$$u'(x) + f(x)u(x) = 0.$$

Suppose one knows a certain solution of the non-homogeneous equation $h(x)$:

$$h' + f(x)h = Q(x),$$

then the general solution of the non-homogeneous equation is:

$$y(x) = Cu(x) + h(x), \quad \forall C \in \mathbb{R}.$$

Method of variable of parameter.

To find a certain solution of the non-homogeneous equation let's try to consider $h(x) = C(x)u(x)$, where $u(x)$ is a solution of the complement homogeneous equation and $C(x)$ is new unknown function.
Substitute the form $h(x)$ into the non-homogeneous equation. It yields:

$$C'u + Cu' + f(x)Cu = Q, \Rightarrow C'u + C(u' + f(x)u) = Q, \Rightarrow C'u = Q(x).$$

Then

$$C' = \frac{Q(x)}{u(x)}, \Rightarrow dC = \frac{Q(x)}{u(x)}dx, \Rightarrow C = \int_{x_0}^{x} \frac{Q(t)}{u(t)}dt.$$

Then a particular solution is: $h(x) = u(x) \int_{x_0}^{x} \frac{Q(t)}{u(t)}dt.$

Theorem about solution of a non-homogeneous linear first-order equation

Theorem

A general solution of the first-order non-homogeneous equation for $y' + f(x) = Q(x)$ can be written in the form:

$$y(x) = Cu(x) + u(x)\int_{x_0}^{x} \frac{Q(t)}{u(t)}dt,$$

where $u(x)$ is a solution of the complementary homogeneous equation $u' + f(x)u = 0$ and x_0 is some constant.

To **proof** this theorem one should differentiate the function $y(x)$ with respect to x.

Non-homogeneous systems

Let's consider the system:

$$Y' = AY + B.$$

Define the fundamental set of solutions for the complimentary system (homogeneous one):

$$U' = AU, \quad \det(U) \neq 0.$$

Denote $Y = U \cdot C(x)$, where $C(x)$ is vector of unknown functions. After substitution of the formula for Y into the equation one gets:

$$U' \cdot C + U \cdot C' = A \cdot U \cdot C + B,$$
$$U \cdot C' + U' \cdot C - A \cdot U \cdot C = B,$$
$$U \cdot C' + (U' - A \cdot U) \cdot C = B.$$

Non-homogeneous systems

Through the non-zero value of the Wronskian for the fundamental set of solutions the inverse matrix of U exists and hence:

$$U \cdot C' = B \Rightarrow C' = U^{-1}B,$$
$$C = \int U^{-1}(x) \cdot B(x)dx.$$

Non-homogeneous system. An example

$$\frac{d}{dx}\mathbf{y} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathbf{y} + \begin{pmatrix} \sin(x) \\ 1 \end{pmatrix}.$$

$$U = \begin{pmatrix} e^x & x\,e^x \\ 0 & e^x. \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} e^{-x} & -x\,e^{-x} \\ 0 & e^{-x}. \end{pmatrix},$$

$$y = \begin{pmatrix} e^x & x\,e^x \\ 0 & e^x. \end{pmatrix} \int \begin{pmatrix} e^{-x} & -x\,e^{-x} \\ 0 & e^{-x}. \end{pmatrix} \begin{pmatrix} \sin(x) \\ 1 \end{pmatrix} dx =$$

$$\begin{pmatrix} e^x & x\,e^x \\ 0 & e^x. \end{pmatrix} \begin{pmatrix} \int e^{-x}\sin(x) - xe^{-x}dx \\ \int e^{-x}dx \end{pmatrix}.$$

Non-homogeneous system. An example

$$y = \begin{pmatrix} e^x & x\,e^x \\ 0 & e^x. \end{pmatrix} \begin{pmatrix} -e^{-x}\left(\frac{1}{2}(\sin(x)+\cos(x)) - (x+1)\right) \\ -e^{-x} \end{pmatrix} + \begin{pmatrix} e^x & x\,e^x \\ 0 & e^x \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix},$$

$$y = \begin{pmatrix} 1 - \frac{1}{2}(\sin(x)+\cos(x)) \\ -1 \end{pmatrix} + \begin{pmatrix} e^x C_1 + x\,e^x C_2 \\ e^x C_2 \end{pmatrix}.$$

Autonomous equations

A system of equations which does not contain the independent variable is called *autonomous system*

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}),$$

The equation for a pendulum is a typical example of the autonomous system:

$$\ddot{\phi} + \sin(\phi) = 0 \Rightarrow \begin{cases} \dot{y}_1 = & y_2, \\ \dot{y}_2 = & -\sin(y_1). \end{cases}$$

Autonomous systems

The predator-pray system is autonomous system:

$$\begin{cases} \dot{y}_1 = & y_1 - y_1 y_2, \\ \dot{y}_2 = & k y_2(-1 + y_1). \end{cases}$$

Any non-autonomous system can be rewritten as autonomous one:

$$\dot{\mathbf{y}} = f(\mathbf{y}, t), \text{ define } y_{n+1} = t \Rightarrow$$
$$\begin{cases} \dot{y}_k = f_k(y_1, \ldots, y_{n+1}), k = 1, \ldots, n; \\ \dot{y}_{n+1} = 1. \end{cases}$$

Phase curves for the pendulum

Let's consider the sum of kinetic and potential energy of the pendulum:

$$E = \frac{\dot{\phi}^2}{2} - \cos(\phi),$$

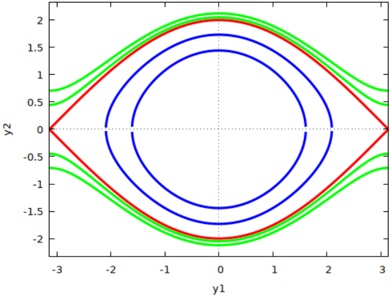
$$\frac{dE}{dt} = \dot{\phi}\ddot{\phi} + \sin(\phi)\dot{\phi} = \dot{\phi}(\ddot{\phi} + \sin(\phi)) = 0.$$

The full energy is a conservation law for the pendulum. These property can be used for defining phase curves.

$$\frac{y_2^2}{2} - \cos(y_1) = E, \quad y_2 = \pm\sqrt{2E + 2\cos(y_1)},$$

$y_2 \in \mathbb{R},\ y_1 \in \mathbb{S} \Rightarrow (y_1, y_2) \in \mathbb{S} \times \mathbb{R}.$

Phase curves for the pendulum



- $E = -1, \ (y_1, y_2) = (0, 0)$;
- $-1 < E < 1, \ y_2 = \pm\sqrt{2E + 2\cos(y_1)}, \ y_1 \in [-\arccos(E), \arccos(E)]$ (blue curves);
- $E = 1, \ (y_1, y_2) \in (0, \pi) \cup y_2 = \pm\sqrt{2E - 2\cos(y_1)}, y_1 \in (-\pi, \pi)$(red curve);
- $1 < E, \ y_2 = \pm\sqrt{2E - 2\cos(y_1)}, \ y_1 \in [-\pi, \pi)$ (green curves).

The conservation law for the predator-prey model

Let us divide the equation

$$\frac{dv}{d\tau} = -k(1-u)v,$$

by the equation

$$\frac{du}{d\tau} = (1-v)u.$$

As a result we obtain:

$$\frac{dv}{du} = \frac{-k(1-u)v}{(1-v)u}.$$

The conservation law for the predator-prey model

Then rewrite the equation in the differential form:

$$(1-v)\frac{dv}{v} = -k(1-u)\frac{du}{u}$$

or

$$\frac{dv}{v} - dv = kdu - k\frac{du}{u}.$$

After integrating we get:

$$\log(v) - v = -k\log(u) + ku + C.$$

The conservation law for the predator-prey model

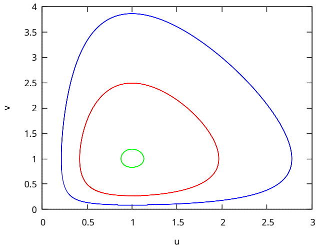


Рис.: The phase portrait of the predator-prey model, $k = 2$.

The value

$$C = \log(vu^k) - (ku + v)$$

is a conservation law for the predator-prey model:

$$\frac{dC}{d\tau} = \frac{dv}{d\tau}\frac{u^k}{vu^k} + k\frac{du}{d\tau}\frac{u^{k-1}v}{vu^k} - k\frac{du}{d\tau} - \frac{dv}{d\tau} = -k(1-u) + k(1-v) - k(1-v)u + k(1-u)v = -k + ku + k - kv - ku + kvu + kv - kuv = 0.$$

Conservation law

The function $U(\mathbf{x})$ is a conservation law of the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

if

$$\sum_{k=1}^{n} \frac{\partial U}{\partial x_k} f_k(\mathbf{x}) = 0.$$

Non-conservative pendulum

Let's consider a pendulum with friction:

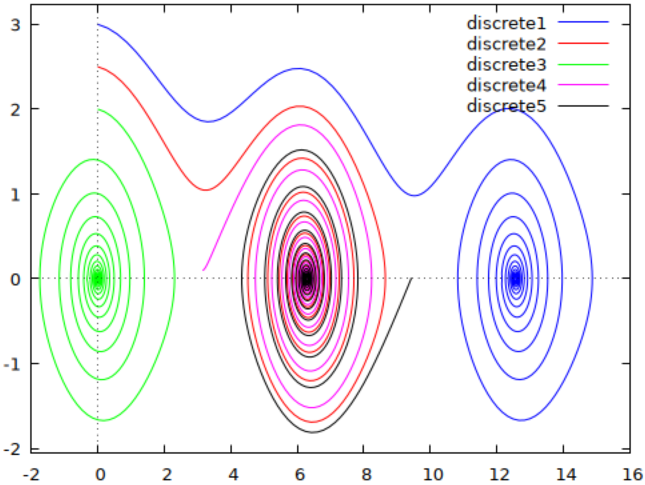
$$\ddot{\phi} + \mu\dot{\phi} + \sin(\phi) = 0 \Rightarrow \begin{cases} y_1' = & y_2, \\ y_2' = & -\mu y_2 - \sin(y_1). \end{cases}$$

Here $\mu > 0$ is a friction coefficient.
Find evolution of the full energy:

$$\frac{dE}{dt} = \dot{\phi}\ddot{\phi} + \sin(\phi)\dot{\phi} = \dot{\phi}(\ddot{\phi} + \sin(\phi)) = -\mu\dot{\phi}^2.$$

The energy of the pendulum with friction decreases.

Non-conservative pendulum



A predator pray system with competing species

$$\frac{dv}{d\tau} = -k(1-u)v - av^2, \frac{du}{d\tau} = (1-v)u - bu^2.$$

The derivative of the conservation law for the predator-pray system:

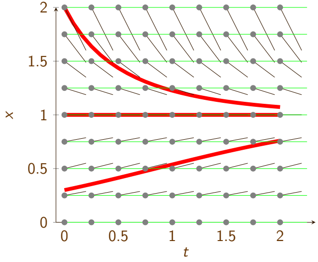
$$C = \log(vu^k) - (ku + v),$$

$$\frac{dC}{d\tau} = \left(\frac{1}{v} - 1\right)\dot{v} + k\left(\frac{1}{v} - 1\right)\dot{u} - k\dot{u} - \dot{v} =$$

$$a(v^2 - v) + bk(u^2 - u).$$

As a result one gets that the C changes under evolution of the system with competing species.

An example. A logistic equation $\dot{x} = (1 - x)x$



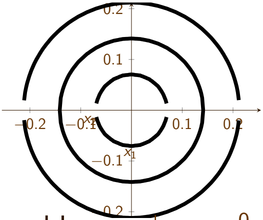
- $x \equiv 1$ is a solution.
- $x(t) : \ x|_{t_0} = x_0$

exists $\forall t > t_0$.

- If $|x_0 - 1| < \epsilon$,

then $\forall t > t_0, \ |x(t) - 1| < \epsilon$.

A mathematical pendulum $\ddot{u} + \sin(u) = 0$



Let's rewrite the equation into the system of equations:

$$\dot{x}_1 = x_2,\ \dot{x}_2 = -\sin(x_1).$$

Define the initial value problem: $x_1|_{t_0} = x_1^0,\ x_2|_{t=t_0} = x_2^0,$

$$\frac{(x_2^0)^2}{2} + (1 - \cos(x_1^0)) < \epsilon < 2$$

Then the solution exists $\forall t > t_0$ and

$$\frac{x_2^2}{2} + (1 - \cos(x_1)) < \epsilon.$$

A definition

Let's consider a solution of the system of equations:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \;\; \mathbf{x} = (x_1(t), x_2(t), \ldots, x_n(t))$$

and assume the solution $\mathcal{X}(t)$ for given initial condition $\mathcal{X}(t_0) = \mathcal{X}^0$ exists for all $t > t_0$.

The solution $\mathcal{X}(t)$ is called stable by Lyapunov

if $\forall \epsilon > 0 \; \exists \delta > 0$, such that $\forall t > t_0$

$$||\mathbf{x}(t) - \mathcal{X}(t)|| < \epsilon$$

for any solution $\mathbf{x}(t)$ such that $||\mathbf{x}(t_0) - \mathcal{X}^0|| < \delta$.

Positive-definite functions

A function $\Phi(\mathbf{x})$ is positive-definite in a manifold $0 \in \mathcal{M}$ if

$$\Phi(\mathbf{x}) > 0,\ \mathbf{x} \neq 0,\ \Phi(0) = 0.$$

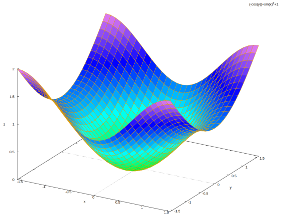
Examples:

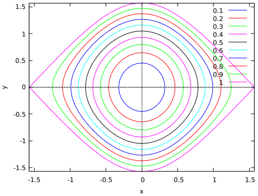
$$\mathbf{1})\ \Phi(x_1, x_2) \equiv x_1^2 + x_2^2,\ \mathbf{x} \in \mathbb{R}^2;$$

$$\mathbf{2})\ \Phi(x_1, x_2) \equiv \sin^2(x_1) + (1 - \cos(x_2)),$$

$$\mathbf{x} \in (-\pi/2, \pi/2) \times (-\pi/2, \pi/2).$$

Contours of given levels





If the function
is positive-definite function
then the the manifolds
of levels $V(\mathbf{x}) = \varepsilon > 0$
are closed curves
around the point $\mathbf{x} = 0$.

Lyapunov's function

Definition

A continuously differentiable function $L(\mathbf{x})$ is called Lyapunov function for the system

$$\dot{\mathbf{x}} = f(\mathbf{x}), \quad f(\mathbf{0}) = 0,$$

at the equilibrium $x = 0$ if:

- $L \in (C^1)$, $L(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$;
- $L(\mathbf{x}) = 0$, $\mathbf{x} = 0$;
- $L(\mathbf{x}) > 0$, $\mathbf{x} \neq 0$;
- $\exists \epsilon > 0$ such that $\dot{L}(\mathbf{x}(t)) \leq 0$, as $\forall ||\mathbf{x}|| < \epsilon$.

Second Lyapunov's stability theorem

Theorem

If there exist the Lyapunov function for the system

$$\dot{\mathbf{x}} = f(\mathbf{x}), \quad f(\mathbf{0}) = 0,$$

then the zero equilibrium is stable.

A scetch of proof. Define $\delta(\epsilon) = diam(L(\mathbf{x}) = \epsilon)$. If $\dot{L} \leq 0$, then the current position should be into or on the contour $L = \epsilon$. Therefore, the conditions of the Lyapunov stability fill.

Asymptotic stability

Definition

The equilibrium **a** is called asymptotic stable if the equilibrium is stable and

$$\exists\delta : ||\mathbf{x}(0)-\mathbf{a}|| < \delta, \Rightarrow \lim_{t\to\infty}\mathbf{x}(t) = a.$$

Theorem about asymptotic stability

Theorem

Let $\dot{L} < 0$, $\mathbf{x} \in \mathcal{M}_\epsilon$, $\partial\mathcal{M}_\epsilon : L(\mathbf{x})$, then $\lim_{t\to\infty} \mathbf{x}(t) = a$.

A scetch of proof.

Suppose $\exists\alpha : \epsilon > \alpha > 0,\ L(\mathbf{x}) \to \alpha$. Then for $L(\mathbf{x}) = \alpha > 0,\ \dot{L} = 0$ it contradicts to the condition of the theorem.

An example. A load with a spring

$$m\ddot{x} + kx = 0,$$

$$\frac{d^2x}{\frac{k}{m}dt^2} + x = 0, \quad \tau = \sqrt{\frac{k}{m}}t,$$

$$\frac{d^2x}{d\tau^2} + x = 0$$

$$\frac{dx_1}{d\tau} = x_2, \ \frac{dx_2}{d\tau} = -x_1.$$

A load with a spring

$$\frac{dx_1}{d\tau} = x_2, \ \frac{dx_2}{d\tau} = -x_1.$$

The full mechanical energy can be considered as a Lyapunov function:

$$L = \frac{x_2^2}{2} + \frac{x_1^2}{2},$$
$$\frac{d}{d\tau}L = x_2\frac{dx_2}{d\tau} + x_1\frac{dx_1}{d\tau} =$$
$$x_2(-x_1) + x_1x_2 = 0.$$

An example. A nonlinear oscillator

$$u'' + u - u^3 = 0.$$

$$u = x_1,\ u' = x_2,$$

$$x_1' = x_2,\ x_2' = -x_1 + x_1^3.$$

Let's examine a function

$$\mathcal{L} = \frac{1}{2}x_2^2 + \frac{1}{2}x_1^2,$$

$$\frac{d}{dt}\mathcal{L} = x_2 x_2' + x_1 x_1' =$$

$$x_2(-x_1 + x_1^3) + x_1 x_2 = x_2 x_1^3$$

So, the function $\mathcal{L}$ cannot be a Lyapunov function for given dynamical system.

An example. A nonlinear oscillator

The Lyapunov function:

$$L \equiv \frac{x_2^2}{2} + \frac{1}{2}x_1^2 - \frac{1}{4}x_1^4.$$

If $x_1^2 + x_2^2 < 1$ then the $L(x_1, x_2) > 0, \ (x_1, x_2) \neq (0, 0)$ and

$$\dot{L} = x_2\dot{x}_2 + \dot{x}_1(x_1 - x^3) =$$
$$x_2(-x_1 + x_1^3) + x_2(x_1 - x_1^3) = 0.$$

Stability of equilibrium

The $\mathbf{a} \in \mathbb{R}^n$ is called equilibrium of the system $\dot{\mathbf{x}} = f(\mathbf{x})$, if $f(\mathbf{a}) \equiv 0$.

First Lyapunov's stability theorem

Let $f(\mathbf{x})$ be differentiable and real parts of all eigenvalues λ_k for the matrix

$$\left.\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{a}} \equiv \left.\left(\frac{\partial f_k}{\partial x_l}\right)\right|_{\mathbf{x}=\mathbf{a}}$$

are negative, then the equilibrium $\mathbf{a}$ is stable solution of the system $\dot{\mathbf{x}} = f(\mathbf{x})$.

The usage of the first Lyapunov's stability theorem

Let's consider the logistic equation.

$$\dot{x} = (1 - x)x.$$

There are two points of equilibrium $x \equiv 0$ and $x \equiv 1$.

$$A \equiv \frac{\partial}{\partial x}(x - x^2) = 1 - 2x,$$
$$A|_{x=1} = -1, \quad |A - \lambda| = 0, \lambda = -1.$$

The equilibrium $x = 1$ is stable.

$$A|_{x=0} = 1, \quad |A - \lambda| = 0, \lambda = 1.$$

Then the equilibrium $x = 0$ does not meet the terms of first Lyapunov's stability theorem.

A pendulum with small viscosity

$$\ddot{u} + \mu\dot{u} + \sin(u) = 0,$$
$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & x_2 \\ -\sin(x_1) & -\mu x_2 \end{pmatrix},$$

The pendulum has two points of equilibrium: $(u, \dot{u}) = (0, 0)$ and $(u, \dot{u}) = (-\pi, 0)$ in the phase space $\mathbb{S} \times \mathbb{R}$.

$$A = \begin{pmatrix} 0 & 1 \\ \cos(x_1) & \mu \end{pmatrix}\Bigg|_{(x_1,x_2)=(0,0)} = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix}.$$
$$-\lambda(-\mu - \lambda) + 1 = 0, \;\; \lambda_{1,2} = \frac{-\mu \pm \sqrt{\mu^2 - 4}}{2}.$$

Then the point $(0, 0)$ is stable.

A pendulum with small viscosity

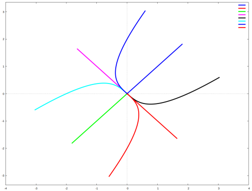
Let's consider the point $(-\pi, 0)$.

$$A = \begin{pmatrix} 0 & 1 \\ \cos(x_1) & \mu \end{pmatrix}\Bigg|_{(x_1,x_2)=(-\pi,0)} = \begin{pmatrix} 0 & 1 \\ 1 & \mu \end{pmatrix}.$$

$$-\lambda(-\mu - \lambda) - 1 = 0, \ \lambda_1\lambda_2 = -1.$$

Then due to the first Lyapunov's stability theorem the point $(-\pi, 0)$ does not meet the terms of the first Lyapunov's stability theorem.

An unstable knot. $\lambda_1 > \lambda 2 > 0$.



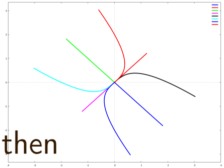
$$\frac{dy}{dt} = 2y + x, \quad \frac{dx}{dt} = y + 2x. \qquad \begin{vmatrix} (2-\lambda) & 1 \\ 1 & (2-\lambda) \end{vmatrix} = 0,$$

$$\lambda_1 = 3,\ \alpha_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ \lambda_2 = 1,\ \alpha_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

A general solution:

$$\begin{pmatrix} y \\ x \end{pmatrix} = C_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

A stable knot. $\lambda_1 < \lambda_2 < 0$



$$\frac{dy}{dt} = -2y + x, \quad \frac{dx}{dt} = y - 2x. \qquad \begin{vmatrix} (-2-\lambda) & 1 \\ 1 & (-2-\lambda) \end{vmatrix} = 0,$$

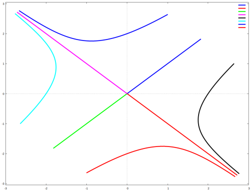
then

$$\lambda_1 = -3,\ \alpha_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix},\ \lambda_2 = -1\ \alpha_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

A general solution:

$$\begin{pmatrix} y \\ x \end{pmatrix} = C_1 e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

A saddle point. $\lambda_2 < 0 < \lambda_1$



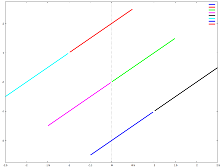
$$\frac{dy}{dt} = y + 2x, \quad \frac{dx}{dt} = 2y + x. \qquad \begin{vmatrix} (1-\lambda) & 2 \\ 2 & (1-\lambda) \end{vmatrix} = 0.$$

$$\lambda_1 = 3,\ \alpha_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ \lambda_2 = -1,\ \alpha_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

A general solution:

$$\begin{pmatrix} y \\ x \end{pmatrix} = C_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

An unstable line. $\lambda_1 = 0,\ \lambda_2 > 0$



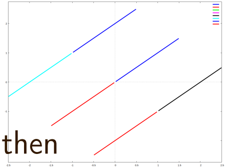
$$\frac{dy}{dt} = y + x, \quad \frac{dx}{dt} = y + x. \qquad \begin{vmatrix} (1-\lambda) & 1 \\ 1 & (1-\lambda) \end{vmatrix} = 0.$$

$$\lambda_1 = 0,\ \alpha_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix},\ \lambda_2 = 2,\ \alpha_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

A general solution:

$$\begin{pmatrix} y \\ x \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

A stable line. $\lambda_1 < 0,\ \lambda_2 = 0$



$$\frac{dy}{dt} = -y - x, \quad \frac{dx}{dt} = -y - x. \qquad \begin{vmatrix} (-1-\lambda) & -1 \\ -1 & (-1-\lambda) \end{vmatrix} = 0,$$

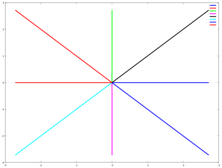
then

$$\lambda_1 = -2,\ \alpha_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ \lambda_2 = 0,\ \alpha_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

A general solution:

$$\begin{pmatrix} y \\ x \end{pmatrix} = C_1 e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

A degenerated stable knot. One eigenvalue and two eigenvectors



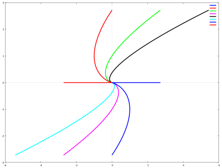
$$\frac{dy}{dt} = y, \quad \frac{dx}{dt} = x. \qquad \begin{vmatrix} (1-\lambda) & 0 \\ 0 & (1-\lambda) \end{vmatrix} = 0,$$

$$\lambda_1 = 1,\ \alpha_1 = e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \lambda_2 = 1,\ \alpha_2 = e^t \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

A general solution:

$$\begin{pmatrix} y \\ x \end{pmatrix} = C_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^t \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

A degenerated unstable knot. Joint vector



$$\frac{dy}{dt} = y + x, \quad \frac{dx}{dt} = x. \qquad \begin{vmatrix} (1-\lambda) & 1 \\ 0 & (1-\lambda) \end{vmatrix} = 0,$$

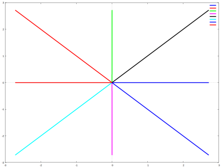
$$\lambda_1 = 1,\ \alpha_1 = e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$\text{joint vector: } \alpha_2 = e^t \begin{pmatrix} t \\ 1 \end{pmatrix}.$$

A general solution:

$$\begin{pmatrix} y \\ x \end{pmatrix} = C_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^t \begin{pmatrix} t \\ 1 \end{pmatrix}.$$

A degenerated stable knot. Two eigenvectors



$$\frac{dy}{dt} = -y, \quad \frac{dx}{dt} = -x. \qquad \begin{vmatrix} (-1-\lambda) & 0 \\ 0 & (-1-\lambda) \end{vmatrix} = 0,$$

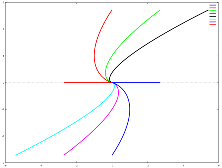
$$\lambda_1 = -1,\ \alpha_1 = e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$\lambda_2 = -1,\ \alpha_2 = e^{-t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

A general solution:

$$\begin{pmatrix} y \\ x \end{pmatrix} = C_1 e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{-t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

A degenerated stable knot. Joint vector



$$\frac{dy}{dt} = -y - 5x, \quad \frac{dx}{dt} = -x. \qquad \begin{vmatrix} (-1-\lambda) & -5 \\ 0 & (-1-\lambda) \end{vmatrix} = 0,$$

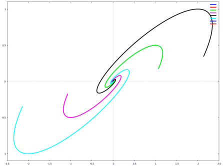
$$\lambda_1 = -1,\ \alpha_1 = e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$\text{joint vector: } \alpha_2 = e^t \begin{pmatrix} t \\ -1/5 \end{pmatrix}.$$

A general solution:

$$\begin{pmatrix} y \\ x \end{pmatrix} = C_1 e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} t \\ -1/5 \end{pmatrix}.$$

A stable focus. $\Re(\lambda_{1,2}) < 0$



$$\frac{dy}{dt} = y - 3x, \quad \frac{dx}{dt} = y - 2x. \qquad \begin{vmatrix} (1-\lambda) & -3 \\ 1 & (-2-\lambda) \end{vmatrix} = 0,$$

$$\lambda_1 = -\frac{1+i\sqrt{3}}{2}, \quad \alpha_1 = e^{\lambda_1 t}\begin{pmatrix} 1 \\ \frac{3+i\sqrt{3}}{6} \end{pmatrix},$$

$$\lambda_2 = \frac{-1+i\sqrt{3}}{2}, \quad \alpha_2 = e^{\lambda_2 t}\begin{pmatrix} 1 \\ \frac{3-i\sqrt{3}}{6} \end{pmatrix}.$$

A general solution:

$$\begin{pmatrix} y \\ x \end{pmatrix} = C_1 e^{\lambda_1 t}\begin{pmatrix} 1 \\ \frac{3+i\sqrt{3}}{6} \end{pmatrix} + C_2 e^{\lambda_2 t}\begin{pmatrix} 1 \\ \frac{3-i\sqrt{3}}{6} \end{pmatrix}.$$

A real-valued solutions

Lemma

Suppose one get complex valued solution of a system with real coefficients. Then the real part of the solution and imaginary part of the solution are solutions of the system.

Proof.

Consider $y = u(t) + iv(t)$, $x(t) = p(t) + iq(t)$, where u, v, p, q are real-valued functions. Substitute the formulas into the system of equations and collect the real and imaginary parts.

Example of real solution

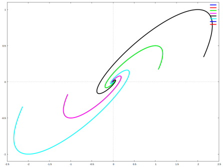
$$
\begin{aligned}
\begin{pmatrix} y_1 \\ x_1 \end{pmatrix} &= e^{-t/2}\Re\left(\left(\cos\left(\frac{\sqrt{3}}{2}t\right) - i\sin\left(\frac{\sqrt{3}}{2}t\right)\right)\begin{pmatrix} 1 \\ \frac{3+i\sqrt{3}}{6} \end{pmatrix}\right) \\
&= e^{-t/2}\begin{pmatrix} \cos\left(\frac{\sqrt{3}}{2}t\right) \\ \frac{1}{2}\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{\sqrt{3}}{6}\sin\left(\frac{\sqrt{3}}{2}t\right) \end{pmatrix}, \\
\begin{pmatrix} y_2 \\ x_2 \end{pmatrix} &= e^{-t/2}\Im\left(\left(\cos\left(\frac{\sqrt{3}}{2}t\right) - i\sin\left(\frac{\sqrt{3}}{2}t\right)\right)\begin{pmatrix} 1 \\ \frac{3+i\sqrt{3}}{6} \end{pmatrix}\right) \\
&= e^{-t/2}\begin{pmatrix} -\sin\left(\frac{\sqrt{3}}{2}t\right) \\ \frac{1}{2}\sin\left(\frac{\sqrt{3}}{2}t\right) + \frac{\sqrt{3}}{6}\cos\left(\frac{\sqrt{3}}{2}t\right) \end{pmatrix}.
\end{aligned}
$$

A general solution

$$\begin{pmatrix} y \\ x \end{pmatrix} = c_1 e^{-t/2} \begin{pmatrix} \cos\left(\frac{\sqrt{3}}{2}t\right) \\ \frac{1}{2}\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{\sqrt{3}}{6}\sin\left(\frac{\sqrt{3}}{2}t\right) \end{pmatrix} + c_2 e^{-t/2} \begin{pmatrix} -\sin\left(\frac{\sqrt{3}}{2}t\right) \\ \frac{1}{2}\sin\left(\frac{\sqrt{3}}{2}t\right) + \frac{\sqrt{3}}{6}\cos\left(\frac{\sqrt{3}}{2}t\right) \end{pmatrix}.$$

Here $c_{1,2} \in \mathbb{R}$.

An unstable focus. $\Re(\lambda_{1,2}) > 0$



$$\frac{dy}{dt} = y - x, \qquad \frac{dx}{dt} = y + \frac{1}{2}x. \qquad \begin{vmatrix} (1-\lambda) & -1 \\ 1 & (\frac{1}{2}-\lambda) \end{vmatrix} = 0,$$

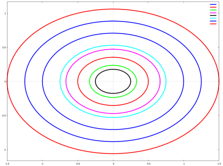
$$\lambda_1 = \frac{3 - i\sqrt{15}}{4}, \;\; \alpha_1 = e^{\lambda_1 t}\begin{pmatrix} 1 \\ \frac{1+i\sqrt{15}}{4} \end{pmatrix},$$

$$\lambda_2 = \frac{3 + i\sqrt{15}}{4}, \;\; \alpha_2 = e^{\lambda_2 t}\begin{pmatrix} 1 \\ \frac{1-i\sqrt{15}}{4} \end{pmatrix}.$$

A general solution:

$$\begin{pmatrix} y \\ x \end{pmatrix} = e^{3t/4}\left(C_1 e^{-i\frac{\sqrt{15}}{4}t}\begin{pmatrix} 1 \\ \frac{1+i\sqrt{15}}{4} \end{pmatrix} + C_2 e^{i\frac{\sqrt{15}}{4}t}\begin{pmatrix} 1 \\ \frac{1-i\sqrt{15}}{4} \end{pmatrix}\right).$$

Center. $\Re(\lambda_{1,2} = 0)$



$$\frac{dy}{dt} = 2x, \quad \frac{dx}{dt} = -y. \qquad \begin{vmatrix} (-\lambda) & 2 \\ -1 & (-\lambda) \end{vmatrix} = 0,$$

$$\lambda_1 = -i\sqrt{2}, \;\; \alpha_1 = e^{-i\sqrt{2}\,t} \begin{pmatrix} 1 \\ \frac{-i}{\sqrt{2}} \end{pmatrix},$$

$$\lambda_2 = i\sqrt{2}, \;\; \alpha_2 = e^{i\sqrt{2}\,t} \begin{pmatrix} 1 \\ \frac{i}{\sqrt{2}} \end{pmatrix}.$$

A general solution:

$$\begin{pmatrix} y \\ x \end{pmatrix} = a \begin{pmatrix} \cos(\sqrt{2}t) \\ -\frac{1}{\sqrt{2}}\sin(\sqrt{2}t) \end{pmatrix} + b \begin{pmatrix} \sin(\sqrt{2}t)) \\ \frac{1}{\sqrt{2}}\cos(\sqrt{2}t) \end{pmatrix}.$$

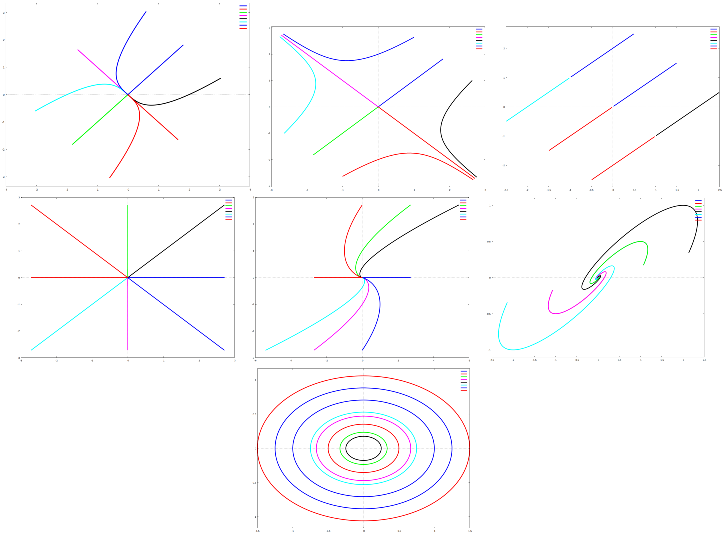
Center. Real-valued solution

$$\begin{pmatrix} y \\ x \end{pmatrix} = r \begin{pmatrix} \cos(\sqrt{2}t + \phi) \\ -\frac{1}{\sqrt{2}} \sin(\sqrt{2}t + \phi) \end{pmatrix},$$

$$r = \sqrt{a^2 + b^2} > 0, \quad \phi = \arctan\left(\frac{b}{a}\right) \in [-\pi/2, \pi/2).$$

$$y^2 + 2x^2 = r^2.$$

Singular points



A stable limit cycle

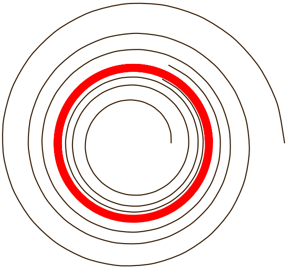


Рис.: The stable limit cycle.

Consider the equation in polar coordinates.

$$\dot{r} = r(1 - r), \quad \dot{\phi} = 1.$$

Then the point $r = 0$ is unstable equilibrium. The point $r = 1$ is a stable one due to the first Lyapunov's theorem the linear part in the neighborhood of the point $r = 1$:

$$r = 1 + R \Rightarrow \dot{R} = -R.$$

A semi-stable limit cycle

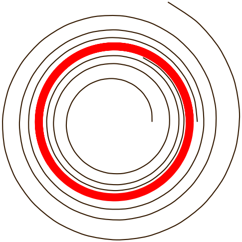


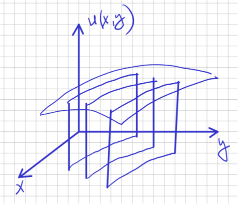
Рис.: The semi-stable limit cycle.

Consider the equation in polar coordinates.

$$\dot{r} = r(r-1)^2, \quad \dot{\phi} = 1.$$

Then the point $r = 0$ is unstable equilibrium. If $r < 1$, then $\dot{r} > 0$, and trajectories tend to $r = 1$. If $r > 1$, then $\dot{r} > 0$, and $r \to \infty$.

A formal approach



Let's consider the equation:

$$\sum_{k=1}^{n} a_k(\vec{x}, u)\partial_{x_k} u = f(u, \vec{x}).$$

and

an initial curve $x_k = y_k(\vec{s}),\ k = 1, \dots, n,$ $u = v(\vec{s}),\ \vec{s} \in \mathbf{R}^{n-1}$. The system of characteristic equations has the form:

$$\frac{dx_k}{dt} = a_k(\vec{x}, u), \quad k = 1, \dots, n, \qquad \frac{du}{dt} = f(u, \vec{x}(t)). \qquad (1)$$

Substitute the characteristic a family of initial conditions:

$$x_k\big|_{t=0} = y_k(\vec{s}), \quad u\big|_{t=0} = v(\vec{s}), \quad \vec{s} \in \mathbf{R}. \qquad (2)$$

The theorem of the existence of unique solution

The function $u(\vec{s}, t)$ is differentiating on s_j, t, then to be differentiation on x_k we need to consider condition for the Jacobian:

$$\frac{\partial(\vec{x})}{\partial(\vec{s}, t)} \neq 0.$$

If the initial curve does not touch of the characteristic curves at $t = 0$, then the following theorem can be formulated.

Theorem

Let the coefficients of the equation and right-hand side are Lipshician on their variables, the initial curve is differentiating on $\vec{s}$ and does not touch to the characteristic curves, then the unique solution exists in a neighborhood of the initial curve.

Counterexamples

An example

$$y\partial_x u - x\partial_y u = 0.$$

The characteristics are the circumstances $x^2 + y^2 = \text{const}$. Let the initial curve be a beam $x > 0$, $y = 1$. On this beam $u = x$, then as $x \to 0, y \to 1$ the equation and initial condition contradict, the problem does not have a solution.
Let's consider the second example for the same equation. If the initial curve coincides to the circumstance $x^2 + y^2 = 1$ and $u = 1$, we do not have a value of the function on others characteristic curves. As a result, we can obtain a lot of different solutions for the same problem:

$$u = x^2 + y^2, \;\; u = (x^2 + y^2)^3.$$

Then the problem does not have a unique solution.

Conservation laws

The derivative on t

$$a(x,y)\partial_x u + b(x,y)\partial_y u = 0$$

gives an opportunity to connect to the system of equations:

$$x' = a(x,y), \quad y' = b(x,y).$$

Then, instead of the equation in partial derivatives, the system of the equations can be considered.
In a general case $u(\vec{x})$ is a solution of the equation

$$\sum_{k=1}^{n} a_k(\vec{x})\partial_{x_k} u = 0. \tag{3}$$

which is a conservation law for the system:

$$\frac{dx_k}{dt} = a_k(\vec{x}), \quad k = 1, \dots, n.$$