First order equations in partial derivatives

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Outlines

An origin of the first-order partial differential equations

The method of characteristics

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Flood waves

Let the wide of a channel be equal to a. The speed of the flow be defined as v and the high of the water in the channel is denoted by $u(x, t)$.

Our aim is to derive an equation for the level of the water in the channel.

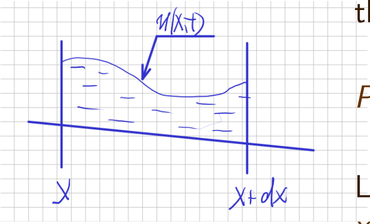
A flow through some section at point x_1 for a time dt can be written as:

$$P_1 = u(x_1)vadt.$$

The same formula defines the flow trough an another section at point x_2 for the same time interval:

$$P_2 = -u(x_2)vadt.$$

Flood waves



It yields
the equation for the water flow

$$P_1+P_2=\int_{x_1}^{x_2}(u(x,t+dt)-u(x,t))vdx,$$

Let's assume
$x_2=x_1+dx$, the one obtains:

$$(avu(x+dx,t)-avu(x,t))dt\sim(u(x,t+dt)-u(x,t))adx$$

Then the limit as $dt, dx\to+0$ leads us to the following equation:

$$\partial_t u-v\partial_x u=0.$$

Optics

Let some surface Γ emits a light. The light spreads along straight lines. Define an imaginary surface reached by the light for time τ as $s(x, y, z)$.
These surface at this moment emits a light beams also and these beams spread by orthogonal direction. The length of the normal interval is proportional to the light speed c. Then:

$$(\partial_x S)^2 + (\partial_y S)^2 + (\partial_z S)^2 = c^2.$$

This equation is called as iconal equation.

Motion of a free particle

$$mx'' = 0.$$

The velocity depends of time and coordinate, then one can consider the function $u = u(x, t)$. In this case the Newtonian equation can be written in the follow form:

$$\frac{du}{dt} = 0,$$

or the same:

$$\partial_t u + \frac{dx}{dt}\partial_x u = 0.$$

And using the definition of the function $u(x, t)$ as the velocity, one gets:

$$\partial_t u + u\partial_x u = 0.$$

Conservation laws

Let be given the system of equations:

$$x' = h_1(x, y, t), \quad y' = h_2(x, y, t). \tag{1}$$

Let's consider the function $u(x, y)$. Suppose the function is differentialbe on the arguments. If $x = x(t)$ and $y = y(t)$, then the derivative on t along the trajectory of the system is:

$$\frac{du}{dt} = h_1 \partial_x u + h_2 \partial_y u.$$

If the derivative equals zero:

$$\frac{du}{dt} \equiv 0,$$

then the function $u(x, y)$ is the *conservation law* of given system (1).

Conservation laws

The derivative on t

$$a(x, y)\partial_x u + b(x, y)\partial_y u = 0$$

gives an opportunity to connect to the system of equations:

$$x' = a(x, y), \quad y' = b(x, y).$$

Then, instead of the equation in partial derivatives, the system of the equations can be considered.
In a general case $u(\vec{x})$ is a solution of the equation

$$\sum_{k=1}^{n} a_k(\vec{x})\partial_{x_k} u = 0. \tag{2}$$

which is a conservation law for the system:

$$\frac{dx_k}{dt} = a_k(\vec{x}), \quad k = 1, \dots, n.$$

Conservation laws. Example

Let's consider the pendulum:

$$u' = v, \quad v' = -sin(u)$$

and the full mechanical energy:

$$E(u, v) = \frac{v^2}{2} - \cos(u).$$

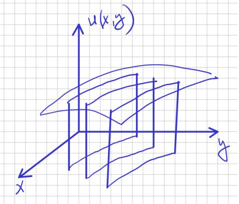
Indeed, one gets:

$$\frac{dE}{dt} = \frac{dv}{dt}\partial_v E + \frac{du}{dt}\partial_u E = v\frac{dv}{dt} + \frac{du}{dt}\sin(u) \equiv 0,$$

then it yields the equation in the partial derivatives:

$$v\partial_u E - \sin(u)\partial_v E = 0.$$

A formal approach



Let's consider the equation:

$$\sum_{k=1}^{n} a_k(\vec{x}, u)\partial_{x_k} u = f(u, \vec{x}).$$

and
an initial curve $x_k = y_k(\vec{s}),\ \ k = 1, \ldots, n$, $u = v(\vec{s})$, $\vec{s} \in \mathbf{R}^{n-1}$. The system of characteristic equations has the form:

$$\frac{dx_k}{dt} = a_k(\vec{x}, u), \ \ k = 1, \ldots, n, \quad \frac{du}{dt} = f(u, \vec{x}(t)). \qquad (3)$$

Substitute the characteristic a family of initial conditions:

$$x_k\big|_{t=0} = y_k(\vec{s}), \quad u\big|_{t=0} = v(\vec{s}), \quad \vec{s} \in \mathbf{R}. \qquad (4)$$

The theorem of the existence of unique solution

The function $u(\vec{s}, t)$ is differentiating on s_j, t, then to be differentiation on x_k we need to consider condition for the Jacobian:

$$\frac{\partial(\vec{x})}{\partial(\vec{s}, t)} \neq 0.$$

If the initial curve does not touch of the characteristic curves at $t = 0$, then the following theorem can be formulated.

Theorem

Let the coefficients of the equation and right-hand side are Lipshician on their variables, the initial curve is differentiating on $\vec{s}$ and does not touch to the characteristic curves, then the unique solution exists in a neighborhood of the initial curve.

Counterexamples

An example

$$y\partial_x u - x\partial_y u = 0.$$

The characteristics are the circumstances $x^2 + y^2 = \text{const}$. Let the initial curve be a beam $x > 0$, $y = 1$. On this beam $u = x$, then as $x \to 0, y \to 1$ the equation and initial condition contradict, the problem does not have a solution.
Let's consider the second example for the same equation. If the initial curve coincides to the circumstance $x^2 + y^2 = 1$ and $u = 1$, we do not have a value of the function on others characteristic curves. As a result, we can obtain a lot of different solutions for the same problem:

$$u = x^2 + y^2, \;\; u = (x^2 + y^2)^3.$$

Then the problem does not have a unique solution.

A area of definite solution

If we know an initial data on some curve then one can find the solution in the area where the characteristic curve, passed through this curve, spread.

Nonlinear equation for the flood waves

Nonlinear equation for the flood waves

Suppose the flow through a sections at x_1 and x_2 are non-linearly connected to the area of the section $A(x, t)$. Let's define as $P(x, t)$ the flow through the section at a point x at the moment t. The area of the section $A(x, t)$ and the flow through the section are connected by the formula

$$P = Av.$$

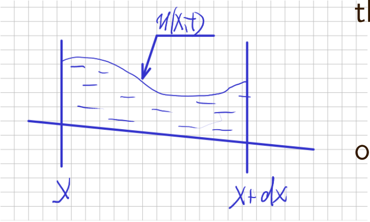
The change of the volume of the water between the sections at x_1 and x_2 for the time $t_2 - t_1$ is:

$$\int_{t_1}^{t_2} P(x_2, t) - P(x_1, t)dt = \int_{x_1}^{x_2} A(x, t_2) - A(x, t_1)dx,$$

Consider the differentials dx, dt:

$$\partial_t A - \partial_x P = 0.$$

Nonlinear equation for the flood waves



Suppose the velocity is a constant, then one gets an equation:

$$\partial_t A - \partial_x (Av) = 0,$$

or the same:

$$\partial_t A - v \partial_x A = 0.$$

Which is different between the flood wave and the speed of flow v. Here the average flow speed is:

$$v = P/A.$$

This speed is defined by gravity and viscosity.

Nonlinear equation for the flood waves

Theoretically the viscosity is proportional to v^2. The equivalence of the gravity and viscosity leads to the equation:

$$Ag\rho\sin(\alpha) = C\rho\Pi v^2.$$

Here ρ is a density, Π is a perimeter of the section, C is viscosity coefficient. Then one gets:

$$v = \sqrt{\frac{Ag\sin(\alpha)}{C\Pi}}, \quad P = Av = \sqrt{\frac{A^3 g\sin(\alpha)}{C\Pi}}.$$

Where $C - \mathrm{const}$, inclination and perimeter Π is supposed to be constants.

Nonlinear equation for the flood waves

Then

$$\partial_t A - v\partial_x A - A\partial_x v = 0 \Rightarrow \partial_t A - (v - \frac{1}{2}v)\partial_x A = 0$$

Then the velocity of the flood wave :

$$\partial_A(vA) = v + A\partial_A v = v + \frac{1}{2}v = \frac{3}{2}v.$$

Conclusion: the speed of the flood wave is faster than the speed of the water flow.

Breaking wave

Breaking wave

Consider the Hopf equation with initial condition

$$\partial_t u + u\partial_x u = 0, \quad u|_{t=0} = u_0(x).$$

The characteristics of the equation are :

$$\frac{dx}{dt} = u_0(s), \quad x|_{t=0} = s.$$

The solution can be written in the parametric form:

$$x = s + u_0(s)t.$$

There are the family of straight lines, inclined as $\tan(\alpha) = u_0(x)$.

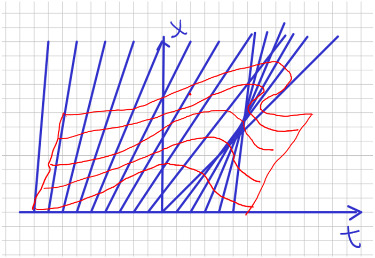
Breaking wave

Let the initial profile be a lump. Then the inclination depends on the initial point for the characteristic straight line. Obviously, that there exists a point (x_*, t_*) at which the characteristics with different inclanation intercects. This point is called the point of wave breaking.
At the wave breaking point the Jacobian for the changing of variables is equal to zero.

$$\frac{\partial(x,t)}{\partial(s,t)} = \begin{vmatrix} 1+u_0't & 0 \\ u_0 & 1 \end{vmatrix} = (1+u_0't) = 0.$$

Hence $t_* = -1/\min(u_0')$, is a moment of the wave breaking.

Breaking wave



The coordinate x of the wave breaking is given by the formula

$$x_* = s_* + u_* t_* .$$

An equation for the curve of the wave break in the parametric form is follows:

$$x = s - \frac{u_0(s)}{u_0'(s)}, \quad s \in \mathbb{R}.$$

An equation for the wavefront

Let's consider the iconal equation:

$$\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 = 1$$

Define: $\partial_x S = p$, $\partial_y S = q$, then the equation has the form:

$$p^2 + q^2 = 1.$$

Differentiate the equation on x:

$$2\frac{\partial^2 S}{\partial x^2}\frac{\partial S}{\partial x} + 2\frac{\partial^2 S}{\partial x \partial y}\frac{\partial S}{\partial y} = 0,$$

Then we get:

$$2p\partial_x p + 2q\partial_y p = 0.$$

Here are two unknown functions p and q.

An equation for the wavefront

To define two unknown function we need to derive one more equation. Differentiating on y we get:

$$2\frac{\partial^2 S}{\partial y \partial x}\frac{\partial S}{\partial x} + 2\frac{\partial^2 S}{\partial y^2}\frac{\partial S}{\partial y} = 0.$$

After substitutions we obtain:

$$2p\partial_x q + 2q\partial_y q = 0.$$

Characteristics for the wavefront

Characteristic curves of these equations coincide:

$$\frac{dx}{dt} = 2p, \quad \frac{dy}{dt} = 2q.$$

The characteristics are beams. Lines for the constant value of S define a wavefront. A tangent line to the wavefront $S = \text{const}$ is defined with the equation:

$$dS = 0, \Rightarrow \partial_x S \frac{dx}{dt} + \partial_y S \frac{dy}{dt} = 0$$

That means the beams are orthogonal to the wavefront.

Characteristics for the wavefront

The equations for p and q on the chracteristic lines are:

$$\frac{dp}{dt} = 0, \quad \frac{dq}{dt} = 0.$$

The solution on these characteristic curves is :

$$x = 2pt + x_0, \quad y = 2qt$$

The equation for S has the form:

$$\frac{dS}{dt} = S_x \frac{dx}{dt} + S_y \frac{dy}{dt} = 2p^2 + 2q^2 = 2(p^2 + q^2) = 2. \Rightarrow S = 2t + S_0,$$

where S_0 is the value S on the initial curve.

A solution for the wavefront

Consider a parabola as an initial curve

$$y^2 = 2ax.$$

Use the parametric form of the curve:

$$y = \alpha, \quad x = \frac{\alpha^2}{2a},$$

then the characteristics are defined with :

$$y = 2qt + \alpha, \quad x = 2pt + \frac{\alpha^2}{2a}.$$

Initial condition is:

$$S = 1, \quad y^2 = 2ax.$$

A solution for the wavefront

On the parabola one gets:

$$\partial_x S \partial_\alpha x + \partial_y S \partial_\alpha y = 0, \Rightarrow p\frac{\alpha}{a} + q = 0, \Rightarrow q = -p\frac{\alpha}{a},$$
$$\Rightarrow p^2 + \left(\frac{\alpha}{a}\right)^2 p^2 = 1, \Rightarrow p^2 = \frac{a^2}{\alpha^2 + a^2}, \quad q^2 = \frac{\alpha^2}{\alpha^2 + a^2}$$

Then:

$$x = -\frac{2at}{\sqrt{\alpha^2 + a^2}} + \frac{\alpha^2}{2a}, \; y = \frac{2\alpha t}{\sqrt{\alpha^2 + a^2}} + \alpha.$$

A solution for the wavefront

The characteristics of the equation intersect when:

$$\frac{\partial(x,y)}{\partial(t,\alpha)} = \begin{vmatrix} -\frac{a}{\sqrt{\alpha^2+a^2}} & \frac{\alpha}{\sqrt{\alpha^2+a^2}} \\ \frac{a\alpha t}{(\sqrt{\alpha^2+a^2})^3} + \frac{\alpha}{a} & -\frac{\alpha^2 t}{(\sqrt{\alpha^2+a^2})^3} + 1 \end{vmatrix} = 0 \Rightarrow$$

$$2a^2 t + (\sqrt{\alpha^2+a^2})^3 = 0 \Rightarrow \frac{2t}{\sqrt{\alpha^2+a^2}} = -\frac{\alpha^2}{a^2} - 1.$$

Then:

$$x = \frac{\alpha^2}{a} + 1 + \frac{\alpha^2}{2a},\ y = -\frac{\alpha^3}{a^2} - \alpha + \alpha \Rightarrow$$

$$a(x-1)\frac{2}{3} = \alpha^2, \quad -a^2 y = \alpha^3, \Rightarrow a^2\frac{8}{27}(x-1)^2 = a^2 y^2,$$

$$\Rightarrow 8(x-1)^3 = 27ay^2.$$

Properties of solution of linear equation

Let's consider the speed of the perturbation in the linear media. Let the initial shape of the wave looks like a lump. Then the higher point of the wave will move on the characteristic curve:

$$\frac{dx}{dt} = -v, \quad x|_{t=0} = x_0.$$

The speed of motion of the wave coincides to the derivative x'. Then the speed of motion of the wave is defined with a coefficient of the derivative $\partial_x u$.

Linear equations in partial derivatives of first order

The equation in the form

$$\sum_{k=1}^{n} a_k(\mathbf{x})\partial_{x_k} u + c(\mathbf{x})u = f(\mathbf{x}, t).$$

is called linear equation in partial derivative of first order.

- The characteristics of the equation do not define on the solution. This allows to define the correctness of the characteristic family and initial curve.
- Due to the linear property a sum of solution is a solution of the same equation.

Semi-linear equations

The equations in the form

$$\sum_{k=1}^{n} a_k(\mathbf{x})\partial_{x_k} u + c(\mathbf{x})u = f(u, \mathbf{x}).$$

Here we suppose the right-hand side contains non-linear dependence of the unknown function u.

- The characteristics of the equation do not define on the solution. This allows to define the correctness of the characteristic family and initial curve.
- Due to the non-linearity a sum of solution is not a solution of the equation.

Quasilinear and non-linear equations

The equations of the form:

$$\sum_{k=1}^{n} a_k(\mathbf{x}, u)\partial_{x_k} u = f(u, \mathbf{x}).$$

are called quasi-linear equations.

- Due to dependence of the coefficients on the solution the correctness of the initial problem depends on the value of the solution on the initial curve.
- Due to the non-linearity a sum of solution is not a solution of the equation.

Other equations are called non-linear equations.

Summary

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