Lyapunov's theory of stability

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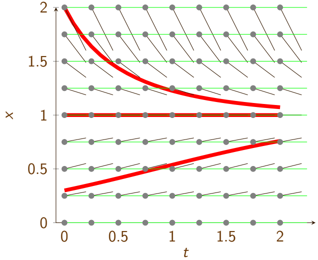
Outlines

An example. A logistic equation $\dot{x} = (1 - x)x$



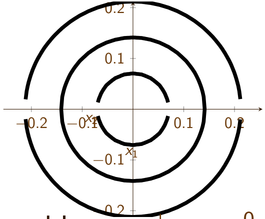
- $x \equiv 1$ is a solution.
- $x(t): \; x|_{t_0} = x_0$

exists $\forall t > t_0$.

- If $|x_0 - 1| < \epsilon$,

then $\forall t > t_0, \; |x(t) - 1| < \epsilon$.

A mathematical pendulum $\ddot{u} + \sin(u) = 0$



Let's rewrite the equation into the system of equations:

$$\dot{x}_1 = x_2,\ \dot{x}_2 = -\sin(x_1).$$

Define the initial value problem: $x_1|_{t_0} = x_1^0,\ x_2|_{t=t_0} = x_2^0,$

$$\frac{(x_2^0)^2}{2} + (1 - \cos(x_1^0)) < \epsilon < 2$$

Then the solution exists $\forall t > t_0$ and

$$\frac{x_2^2}{2} + (1 - \cos(x_1)) < \epsilon.$$

A definition

Let's consider a solution of the system of equations:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}),\ \mathbf{x} = (x_1(t), x_2(t), \dots, x_n(t))$$

and assume the solution $\mathcal{X}(t)$ for given initial condition $\mathcal{X}(t_0) = \mathcal{X}^0$ exists for all $t > t_0$.

The solution $\mathcal{X}(t)$ is called stable by Lyapunov

if $\forall \epsilon > 0\ \exists \delta > 0$, such that $\forall t > t_0$

$$||\mathbf{x}(t) - \mathcal{X}(t)|| < \epsilon$$

for any solution $\mathbf{x}(t)$ such that $||\mathbf{x}(t_0) - \mathcal{X}^0|| < \delta$.

Positive-definite functions

A function $\Phi(\mathbf{x})$ is positive-definite in a manifold $0 \in \mathcal{M}$ if

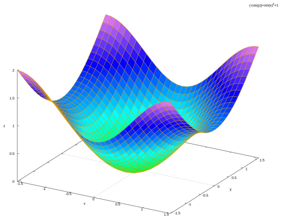
$$\Phi(\mathbf{x}) > 0,\ \ \mathbf{x} \neq 0,\ \ \Phi(0) = 0.$$

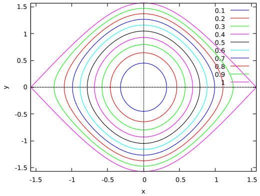
Examples:

$$\mathbf{1})\ \Phi(x_1, x_2) \equiv x_1^2 + x_2^2,\ \ \mathbf{x} \in \mathbb{R}^2;$$

$$\mathbf{2})\ \Phi(x_1, x_2) \equiv \sin^2(x_1) + (1 - \cos(x_2)),$$
$$\mathbf{x} \in (-\pi/2, \pi/2) \times (-\pi/2, \pi/2).$$

Contours of given levels





If the function
is positive-definite function
then the the manifolds
of levels $V(\mathbf{x}) = \varepsilon > 0$
are closed curves
around the point $\mathbf{x} = 0$.

Lyapunov function

Definition

A continuously differentiable function $L(\mathbf{x})$ is called Lyapunov function for the system

$$\dot{\mathbf{x}} = f(\mathbf{x}), \quad f(\mathbf{0}) = 0,$$

at the equilibrium $x = 0$ if:

- $L \in (C^1),\ L(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$;
- $L(\mathbf{x}) = 0,\ \mathbf{x} = 0$;
- $L(\mathbf{x}) > 0,\ \mathbf{x} \neq 0$;
- $\exists \epsilon > 0$ such that $\dot{L}(\mathbf{x}(t)) \leq 0$, as $\forall ||\mathbf{x}|| < \epsilon$.

Second Lyapunov's stability theorem

Theorem

If there exist the Lyapunov function for the system

$$\dot{\mathbf{x}} = f(\mathbf{x}), \quad f(\mathbf{0}) = 0,$$

then the zero equilibrium is stable.

A scetch of proof. Define $\delta(\epsilon) = diam(L(\mathbf{x}) = \epsilon)$. If $\dot{L} \leq 0$, then the current position should be into or on the contour $L = \epsilon$. Therefore, the conditions of the Lyapunov stability fill.

Asymptotic stability

Definition

The equilibrium **a** is called asymptotic stable if the equilibrium is stable and

$$\exists\delta : ||\mathbf{x}(0) - \mathbf{a}|| < \delta, \Rightarrow \lim_{t\to\infty} \mathbf{x}(t) = a.$$

Theorem about asymptotic stability

Theorem

Let $\dot{L} < 0$, $\mathbf{x} \in \mathcal{M}_\epsilon$, $\partial\mathcal{M}_\epsilon : L(\mathbf{x})$, then $\lim_{t\to\infty} \mathbf{x}(t) = a$.

A scetch of proof.

Suppose $\exists\alpha : \epsilon > \alpha > 0,\ L(\mathbf{x}) \to \alpha$. Then for $L(\mathbf{x}) = \alpha > 0,\ \dot{L} = 0$ it contradicts to the condition of the theorem.

An example. A load with a spring

$$m\ddot{x} + kx = 0,$$

$$\frac{d^2x}{\frac{k}{m}dt^2} + x = 0, \quad \tau = \sqrt{\frac{k}{m}}t,$$

$$\frac{d^2x}{d\tau^2} + x = 0$$

$$\frac{dx_1}{d\tau} = x_2, \ \frac{dx_2}{d\tau} = -x_1.$$

A load with a spring

$$\frac{dx_1}{d\tau} = x_2, \;\; \frac{dx_2}{d\tau} = -x_1.$$

The full mechanical energy can be considered as a Lyapunov function:

$$L = \frac{x_2^2}{2} + \frac{x_1^2}{2},$$

$$\frac{d}{d\tau}L = x_2\frac{dx_2}{d\tau} + x_1\frac{dx_1}{d\tau} = x_2(-x_1) + x_1 x_2 = 0.$$

An example. A nonlinear oscillator

$$u'' + u - u^3 = 0.$$
$$u = x_1,\ u' = x_2,$$
$$x_1' = x_2,\ x_2' = -x_1 + x_1^3.$$

Let's examine a function

$$\mathcal{L} = \frac{1}{2}x_2^2 + \frac{1}{2}x_1^2,$$
$$\frac{d}{dt}\mathcal{L} = x_2 x_2' + x_1 x_1' =$$
$$x_2(-x_1 + x_1^3) + x_1 x_2 = x_2 x_1^3$$

So, the function $\mathcal{L}$ cannot be a Lyapunov function for given dynamical system.

An example. A nonlinear oscillator

The Lyapunov function:

$$L \equiv \frac{x_2^2}{2} + \frac{1}{2}x_1^2 - \frac{1}{4}x_1^4.$$

If $x_1^2 + x_2^2 < 1$ then the $L(x_1, x_2) > 0, \ (x_1, x_2) \neq (0, 0)$ and

$$\dot{L} = x_2\dot{x}_2 + \dot{x}_1(x_1 - x^3) =$$
$$x_2(-x_1 + x_1^3) + x_2(x_1 - x_1^3) = 0.$$

Chetaev's theorem about instability

Theorem

Let $\mathcal{M}$ is neighborhood of equilibrium $\mathbf{a}$ and $\mathcal{M}_1 \subset \mathcal{M}$, $\mathbf{a} \in \partial\mathcal{M}_1$, $V(\mathbf{x})$ is continuously differential function and $V(\mathbf{x}) = 0,\ \mathbf{x} \in \partial\mathcal{M}_1$:

$$V(\mathbf{x}) > 0.\ \dot{V} > 0,\ \mathbf{x} \in \mathcal{M}_1.$$

then $\mathbf{a}$ is unstable.

A scatch of proof.

The trajectory avoids the $\partial\mathcal{M}_1$.

An example

$$\dot{x} = x^3 - y, \;\; \dot{y} = x + y^3,$$
$$L = \frac{1}{2}x^2 + \frac{1}{2}x^2,$$
$$\dot{L} = x(x^3 - y) + y(x + y^3),$$
$$\dot{L} = x^4 + y^4.$$

The equilibrium $(0, 0)$ is unstable.

Stability of equilibrium

The $\mathbf{a} \in \mathbb{R}^n$ is called equilibrium of the system $\dot{\mathbf{x}} = f(\mathbf{x})$, if $f(\mathbf{a}) \equiv 0$.

First Lyapunov's stability theorem

Let $f(\mathbf{x})$ be differentiable and real parts of all eigenvalues λ_k for the matrix

$$\left.\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{a}} \equiv \left.\left(\frac{\partial f_k}{\partial x_l}\right)\right|_{\mathbf{x}=\mathbf{a}}$$

are negative, then the equilibrium $\mathbf{a}$ is stable solution of the system $\dot{\mathbf{x}} = f(\mathbf{x})$.

The usage of the first Lyapunov's stability theorem

Let's consider the logistic equation.

$$\dot{x} = (1 - x)x.$$

There are two points of equilibrium $x \equiv 0$ and $x \equiv 1$.

$$A \equiv \frac{\partial}{\partial x}(x - x^2) = 1 - 2x,$$
$$A|_{x=1} = -1, \quad |A - \lambda| = 0, \lambda = -1.$$

The equilibrium $x = 1$ is stable.

$$A|_{x=0} = 1, \quad |A - \lambda| = 0, \lambda = 1.$$

Then the equilibrium $x = 0$ does not meet the terms of first Lyapunov's stability theorem.

A pendulum with small viscosity

$$\ddot{u} + \mu \dot{u} + \sin(u) = 0,$$
$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & x_2 \\ -\sin(x_1) & -\mu x_2 \end{pmatrix},$$

The pendulum has two points of equilibrium: $(u, \dot{u}) = (0, 0)$ and $(u, \dot{u}) = (-\pi, 0)$ in the phase space $\mathbb{S} \times \mathbb{R}$.

$$A = \begin{pmatrix} 0 & 1 \\ \cos(x_1) & \mu \end{pmatrix}\Bigg|_{(x_1,x_2)=(0,0)} = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix}.$$
$$-\lambda(-\mu - \lambda) + 1 = 0, \;\; \lambda_{1,2} = \frac{-\mu \pm \sqrt{\mu^2 - 4}}{2}.$$

Then the point $(0, 0)$ is stable.

A pendulum with small viscosity

Let's consider the point $(-\pi, 0)$.

$$A = \begin{pmatrix} 0 & 1 \\ \cos(x_1) & \mu \end{pmatrix}\Bigg|_{(x_1,x_2)=(-\pi,0)} = \begin{pmatrix} 0 & 1 \\ 1 & \mu \end{pmatrix}.$$

$$-\lambda(-\mu-\lambda) - 1 = 0,\ \ \lambda_1\lambda_2 = -1.$$

Then due to the first Lyapunov's stability theorem the point $(-\pi, 0)$ does not meet the terms of the first Lyapunov's stability theorem.

A stable limit cycle

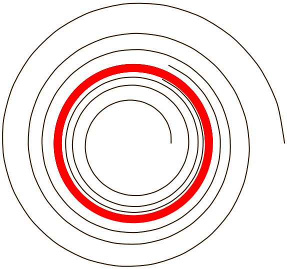


Figure: The stable limit cycle.

Consider the equation in polar coordinates.

$$\dot{r} = r(1-r), \quad \dot{\phi} = 1.$$

Then the point $r = 0$ is unstable equilibrium. The point $r = 1$ is a stable one due to the first Lyapunov's theorem the linear part in the neighborhood of the point $r = 1$:

$$r = 1 + R \Rightarrow \dot{R} = -R.$$

A semi-stable limit cycle

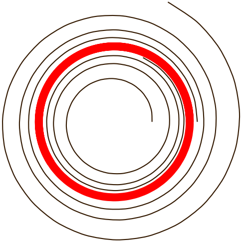


Figure: The semi-stable limit cycle.

Consider the equation in polar coordinates.

$$\dot{r} = r(r-1)^2, \quad \dot{\phi} = 1.$$

Then the point $r = 0$ is unstable equilibrium. If $r < 1$, then $\dot{r} > 0$, and trajectories tend to $r = 1$. If $r > 1$, then $\dot{r} > 0$, and $r \to \infty$.

Limit sycle

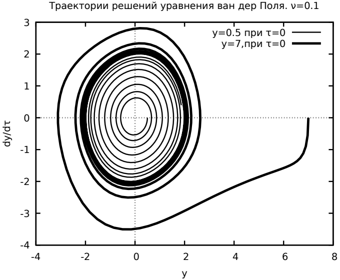
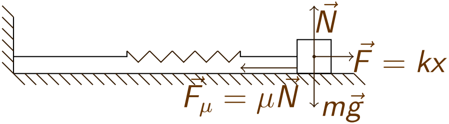


Figure: The limit cycle for the van der Pol equation $\ddot{y} - \nu(1-y^2)\dot{y} + y = 0$.

A **limit cycle** is a closed trajectory which is a limit at least for one other trajectory.

Oscillator with dry friction



If $\dot{x} \neq 0$ then movement of a load with a spring is defined by the following equation:

$$m\ddot{x} = -\mu mg\,\mathrm{sign}(\dot{x}) - kx.$$

If $\dot{x} = 0$, then:

$$m\ddot{x} \in (-\mu mg + kx, \mu mg + kx).$$

Oscillator with dry friction

Formally it means:

$$m\ddot{x} \in \begin{cases} -\mu mg \operatorname{sign}(\dot{x}) - kx, & \dot{x} \neq 0; \\ (-\mu mg + kx, \mu mg + kx), & \dot{x} = 0. \end{cases}$$

Such model is a differential inclusion. The differential inclusion allows us to consider the set $x \in (-\mu mg/k, \mu mg/k)$ and $\dot{x} = 0$ as equilibrium.

An example. A pendulum with dry friction

$$\ddot{u} + \mu\,\mathrm{sign}(\dot{u}) + \sin(u) = 0,\ \ \mu > 0,$$
$$\dot{x}_1 = x_2,\ \ \dot{x}_2 = -\mu\,\mathrm{sign}(x_2) - \sin(x_1).$$

The Lyapunov function:

$$L(x_1, x_2) = \frac{1}{2}x_2^2 + (1 - \cos(x_1)),\ \ L(0,0) = 0,$$
$$\dot{L} = x_2\dot{x}_2 + \dot{x}_1\sin(x_1) =$$
$$x_2(-\mu\,\mathrm{sign}(x_2) - \sin(x_1)) + x_2\sin(x_1) = -\mu\,x_2\,\mathrm{sign}(x_2).$$

Therefore, the equilibrium $(0,0)$ is stable.

Dynamics of the oscillator with dry friction

Consider for simplicity the case $\mathrm{sign}(\dot{x}) \neq 0$. Let us define $\sqrt{k}t/\sqrt{m} = \tau$. It yields:

$$x'' = -x - f\,\mathrm{sign}\left(x'\right), \quad f = \frac{\mu m g}{k}.$$

Then the equation for the movement as $x' > 0$ looks like:

$$x'' = -x - f.$$

Dynamics of the oscillator with dry friction

Multiply the equation by x' then:

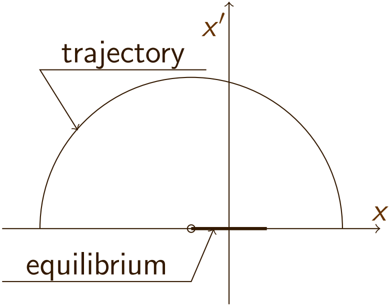
$$x'x'' + x'x = -fx'.$$

This equation we can rewrite as follow:

$$\left(\frac{(x')^2}{2} + \frac{x^2}{2}\right)' = -fx'.$$

This formula shows that the sum in the left-hand side decreases for $x' > 0$.

Dynamics of the oscillator with dry friction



Integration on x yields:

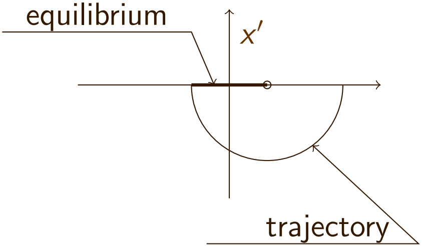
$$\frac{(x')^2}{2} + \frac{x^2}{2} = -fx + E.$$

Here E
is a constant of integration.
We rewrite
previous formula in the form:

$$(x')^2 + (x + f)^2 = f^2 + 2E.$$

That means the trajectory is a semi-circle with the center at $(-f, 0)$ and radius $R = \sqrt{f^2 + 2E}$.

Dynamics of the oscillator with dry friction



In case $\text{sign}(x') < 0$ we obtain:

x $\qquad x'' = -x + f.$

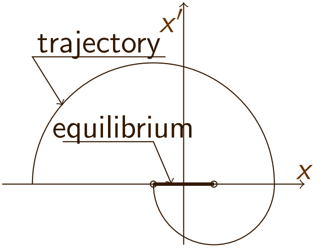
After multiplying by x' and integrating we can rewrite as follow: $\left(\frac{(x')^2}{2} + \frac{x^2}{2}\right)' = fx'.$ So, the sum in the left-hand side decreases for $x' < 0$ also.

After integrating we obtain:

$$(x')^2 + (x - f)^2 = f^2 + 2E_1,$$

That means the trajectory is a semi-circle with the center at $(f, 0)$ and radius $R = \sqrt{f^2 + 2E_1}$.

Dynamics of the oscillator with dry friction

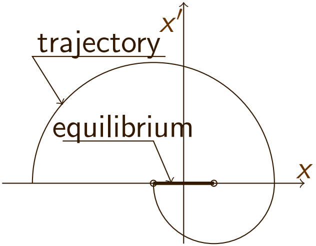


Let the
initial point of the trajectory be
$(x, x') = (x_0, 0)$ where $x_0 < -f$.
The
part of the trajectory for $x' > 0$
is the semicircle with center at
$(-f, 0)$ and radius $r = -f - x_0$.
Then the right point
of this semicircle $(x_1, 0)$, where

$x_1 = x_0 + 2(-f - x_0) = -2f - x_0$

If $x_1 > f$ then this point is initial one for the lower semicircle with left point $(x_1, 0)$ and center at $(f, 0)$ and radius $r = (x_1 - f)$.

Dynamics of the oscillator with dry friction



The left point for this lower semicircle is $(x_2, 0)$, where $x_2 = x_1 - 2(x_1 - f)$,

$x_2 = -x_1 + 2f$,

$x_2 = -(-2f - x_0) + 2f$,

$x_2 = x_0 + 4f$.

If $-f \leq x_2$ then the point $(x_2, 0)$ is equilibrium.

In opposite case the point $(x_0 + 4f, 0)$ is beginning of the next circle. This next circle begins closer to the equilibrium state then the first circle at $(x_0, 0)$.

As the result we get the sequence $\{x_n\}_{k=0}^{n}$ until $x_n \in [-f, f]$.

Summary