First Lyapunov stability theorem

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First Lyapunov stability theorem. Examples

The proof of the first Lyapunov stability theorem

The Lorenz system

Summary

A definition

Let's consider a solution of the system of equations:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}),\ \mathbf{x} = (x_1(t), x_2(t), \ldots, x_n(t))$$

and assume the solution $\mathcal{X}(t)$ for given initial condition $\mathcal{X}(t_0) = \mathcal{X}^0$ exists for all $t > t_0$.

The solution $\mathcal{X}(t)$ is called stable by Lyapunov

if $\forall \epsilon > 0\ \exists \delta > 0$, such that $\forall t > t_0$

$$||\mathbf{x}(t) - \mathcal{X}(t)|| < \epsilon$$

for any solution $\mathbf{x}(t)$ such that $||\mathbf{x}(t_0) - \mathcal{X}^0|| < \delta$.

Stability of equilibrium

The $\mathbf{a} \in \mathbb{R}^n$ is called equilibrium of the system $\dot{\mathbf{x}} = f(\mathbf{x})$, if $f(\mathbf{a}) \equiv 0$.
It is useful to use a new unknown function $\mathbf{y} = \mathbf{x} - \mathbf{a}$. Then the system is transform to the form:

$$\dot{\mathbf{y}} = g(\mathbf{y}),\, g(0) = 0,$$
$$g(\mathbf{y}) \equiv f(\mathbf{y} + \mathbf{a}).$$

So, let's redefine:

$$\mathbf{y} \to \mathbf{x},\; g(\mathbf{y})) \to f(\mathbf{x}),$$

and without loss of a generality we will consider below the function $f(\mathbf{x})$ and the equilibrium $\mathbf{x} = 0$: $f(0) = 0$.

First Lyapunov's stability theorem

Let $f(\mathbf{x})$ be differentiable and real parts of all eigenvalues λ_k for the matrix

$$A = \left.\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}=0} \equiv \left.\left(\frac{\partial f_k}{\partial x_l}\right)\right|_{\mathbf{x}=0}$$

are negative, then the equilibrium $\mathbf{x} = 0$ is stable solution of the system $\dot{\mathbf{x}} = f(\mathbf{x})$.

One dimension case

Let the right-hand side of the equation be following:

$$f(x) = \lambda x + x^2(1 + g(x)),\ \lambda < 0, g(x) \in \mathbf{C}^1[-\Delta, \Delta].$$

The shorter proof. Consider a time derivative of the function $L(x) = x^2$ due to the chain rule:

$$\dot{L}(x) = 2x\dot{x} = 2\lambda x^2 + x^3(1 + g(x)).$$

Then $\exists \epsilon > 0, \forall x :\ x^2 < \epsilon, \dot{L} < 0$.
Due to the second Lyapunov stability theorem the point $x = 0$ is a stable equilibrium.

A two-dimensional case. Two real eigenvalues

$$
\begin{aligned}
f_1(x_1,x_2) &= x_1 + x_2 + O(\mathbf{x}^2),\\
f_1(x_1,x_2) &= 2x_2 + O(\mathbf{x}^2).\\
A &= \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix},\ \lambda_1 = -1,\ \lambda_2 = -2.\\
T &= \begin{pmatrix} 1 & b \\ 0 & -b \end{pmatrix}, \quad T^{-1}AT = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}.
\end{aligned}
$$

A two-dimensional case. Two real eigenvalues

$$
\begin{aligned}
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 1 & b \\ 0 & -b \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \\
\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} &= \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + O(\mathbf{y}^2). \\
L(\mathbf{y}) &= y_1^2 + y_2^2, \\
\dot{L} &= -y_1^2 - 4y_2^2 + O(\mathbf{y}^3).
\end{aligned}
$$

One eigenvalue of second order

$$\begin{aligned} f_1(x_1, x_2) &= -x_1 + x_2 + O(\mathbf{x}^2), \\ f_1(x_1, x_2) &= \quad\quad -x_2 + O(\mathbf{x}^2). \\ A &= \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, \ \lambda_1 = -1. \\ T &= \begin{pmatrix} 1 & 1 \\ 0 & d \end{pmatrix}, \quad T^{-1}AT = \begin{pmatrix} -1 & d \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

One eigenvalue of second order

$$\begin{aligned}
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 0 & d \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \\
\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} &= \begin{pmatrix} -1 & d \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + O(\mathbf{y}^2). \\
L(\mathbf{y}) &= y_1^2 + y_2^2, \\
\dot{L} &= -y_1^2 + dy_1y_2 - y_2^2 + O(\mathbf{y}^3).
\end{aligned}$$

$$\exists d : -y_1^2 + dy_1y_2 - y_2^2 < 0, \ \forall (y_1, y_2) \in \mathbb{R}^2.$$

Two complex conjugated eigenvalues

$$\begin{aligned} f_1(x_1,x_2) &= x_2 + O(\mathbf{x}^2), \\ f_1(x_1,x_2) &= -x_1 - x_2 + O(\mathbf{x}^2). \\ A &= \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix},\ \lambda_1 = -\frac{1+i\sqrt{3}}{2},\ \lambda_2 = \frac{-1+i\sqrt{3}}{2}. \\ T &= \begin{pmatrix} 2 & -1-i\sqrt{3} \\ 2 & -1+i\sqrt{3} \end{pmatrix}, \quad T^{-1}AT = \begin{pmatrix} -\frac{1+i\sqrt{3}}{2} & 0 \\ 0 & \frac{-1+i\sqrt{3}}{2} \end{pmatrix}. \end{aligned}$$

Two complex conjugated eigenvalues

$$
\begin{aligned}
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 2 & -1 - i\sqrt{3} \\ 2 & -1 + i\sqrt{3} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \\
\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} &= \begin{pmatrix} -\frac{1+i\sqrt{3}}{2} & 0 \\ 0 & \frac{-1+i\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + O(\mathbf{y}^2). \\
L(\mathbf{y}) &= y_1\overline{y}_1 + y_2\overline{y}_2, \\
\dot{L} &= -\frac{1+i\sqrt{3}}{2}|y_1|^2 - \frac{1-i\sqrt{3}}{2}|y_1|^2 - \\
&\quad \frac{1-i\sqrt{3}}{2}|y_2|^2 - \frac{1+i\sqrt{3}}{2}|y_2|^2 + O(\mathbf{y}^3), \\
\dot{L} &= -|y_1|^2 - |y_2|^2 + O(\mathbf{y}^3).
\end{aligned}
$$

The multidimensional case

Let the $f(\mathbf{x})$ be such that $f(\mathbf{x}) \in \mathbf{C}^1$. Define a matrix:

$$A \equiv \left(\frac{\partial f_k}{\partial x_l}\right)\bigg|_{\mathbf{x}=0}.$$

Suppose $\lambda_k,\ k = 1, 2, \ldots, l,\ \ l \leq n$ are given order eigenvalues of the matrix A.
Then $\exists T$ such that:

$$T^{-1}AT = \mathrm{diag}(\Lambda) + B_\epsilon,$$

where $\mathrm{diag}(\Lambda)$ is diagonal matrix, where $\Lambda = \{\lambda_1, \ldots, \lambda_l\}$ are the diagonal elements taken with their orders and B_ϵ is a nilpotent matrix with one sup-diagonal coefficients less than given ϵ.

Lyapunov function

Define $\mathbf{x} = T\mathbf{y}$, then

$$\frac{d}{dt}\mathbf{y} = (\mathrm{diag}(\Lambda) + B_\epsilon)\mathbf{y} + O(\mathbf{y}^2).$$

$$L(\mathbf{y}) = (\mathbf{y}, \bar{\mathbf{y}}),$$

then

$$\frac{d}{dt}L(\mathbf{y}) = (\dot{\mathbf{y}}, \bar{\mathbf{y}}) + (\mathbf{y}, \dot{\bar{\mathbf{y}}}) = 2\sum_k \Re(\lambda_k + b_k)|y_k|^2 + O(\mathbf{y}^3).$$

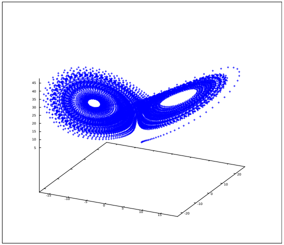
Define $\lambda = \min_k(|\Re(\lambda_k)|)$, then

$$\frac{d}{dt}L(\mathbf{y}) < -(\lambda - \epsilon)L(\mathbf{y}).$$

Then $\mathbf{y} = 0$ is stable equilibrium, hence $\mathbf{x} = 0$ is stable equilibrium also.

The Lorenz system

Origin:



$$\frac{\partial v}{\partial t} + v\nabla v = -\frac{\nabla p}{\rho} + \mu\Delta v + g,$$

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right),$$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

First coefficients of the Fourier series yield:

$$\dot{x} = \sigma(y - x),\ \ \dot{y} = x(\rho - z) - y,\ \ \dot{z} = xy - \beta z.$$

Here β, σ, ρ are parameters of the mathematical model.

Properties of the solution

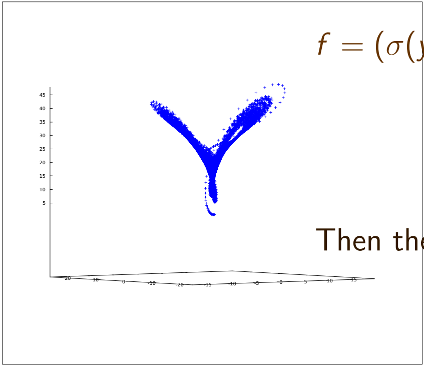
Change the variable:$z = \rho - w$,

$$
\begin{aligned}
\dot{x} &= \sigma(y - x), \quad \times x/\sigma \\
\dot{y} &= xw - y, \quad \times y \\
\dot{w} &= xy - \beta w + \beta\rho. \quad \times w, \\
\frac{1}{2}\frac{d}{dt}(\frac{x^2}{\sigma} + y^2 + w^2) &= xy - x^2 - y^2 - \beta w^2 + \beta\rho, \\
\frac{1}{2}\frac{d}{dt}(\frac{x^2}{\sigma} + y^2 + z^2) &= -(x - \frac{y}{2})^2 - \frac{3}{4}y^2 - \beta(w - \frac{\rho}{2})^2 + \frac{1}{4}\beta\rho^2.
\end{aligned}
$$

Corollary. All trajectories tend into the ellipsoid:

$$(x - \frac{y}{2})^2 + \frac{3}{4}y^2 + \beta(w - \frac{\rho}{2})^2 = \frac{1}{4}\beta\rho^2.$$

The divergence

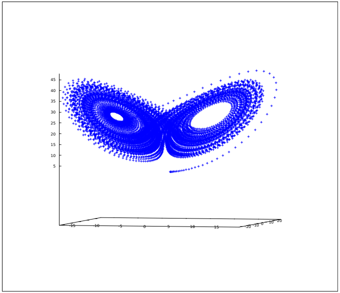


Consider a vector field:

$$f = (\sigma(y-x), x(\rho - z) - y, xy - \beta z),$$
$$\operatorname{div}(f) \equiv \frac{\partial}{\partial x} f_1 + \frac{\partial}{\partial y} f_2 + \frac{\partial}{\partial z} f_3,$$
$$\operatorname{div}(f) = -\sigma - 1 - \beta.$$

Then the phase space contracts.

Existence of attractor

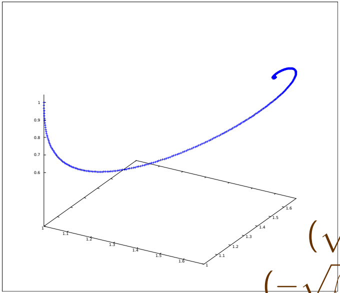


Corollary

One or several attractors are contained in the ellipsoid:

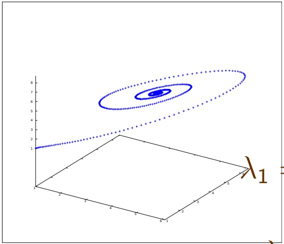
$$(x - \frac{y}{2})^2 + \frac{3}{4}y^2 + \beta(w - \frac{\rho}{2})^2 = \frac{1}{4}\beta\rho^2.$$

Stationary points



$$\sigma(y-x)=0,$$
$$x(\rho-z)-y=0,$$
$$xy-\beta z=0.$$
$$(0,0,0)$$
$$(\sqrt{\beta(\rho-1)},\sqrt{\beta(\rho-1)},\rho-1),$$
$$(-\sqrt{\beta(\rho-1)},-\sqrt{\beta(\rho-1)},\rho-1).$$

Behaviour at $(0,0,0)$



$$A = \begin{pmatrix} -\sigma & \rho & 0 \\ \sigma & -1 & 0 \\ 0 & 0 & -\beta \end{pmatrix}$$

$$\lambda_1 = -\frac{\sqrt{\sigma^2 + (4\rho - 2)\,\sigma + 1} + \sigma + 1}{2},$$

$$\lambda_2 = \frac{\sqrt{\sigma^2 + (4\rho - 2)\,\sigma + 1} - \sigma - 1}{2},$$

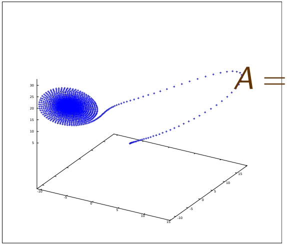
$$\lambda_3 = -\beta.$$

The stability condition:

$$(\sigma + 1) > \sqrt{\sigma^2 + (4\rho - 2)\sigma + 1}$$

$$2 > 4\rho - 1, \Rightarrow \rho < 1.$$

Behaviour at $((\sqrt{\beta(\rho-1)}, \sqrt{\beta(\rho-1)}, \rho-1)$



$$A = \begin{pmatrix} -\sigma & 1 & \sqrt{\beta(\rho-1)} \\ \sigma & -1 & \sqrt{\beta(\rho-1)} \\ 0 & -\sqrt{\beta(\rho-1)} & -\beta \end{pmatrix}$$

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Pictures