Nonlinear system and numerical approach

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Equilibrium point

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A predator pray system with competing species

$$\begin{cases} \dot{v} = -k(1-u)v - av^2, \\ \dot{u} = (1-v)u - bu^2. \end{cases}$$

When $v = 0$, we get a logistic equation that determines a reproduction of carps in a pond with a limited feed.
Pikes without carps will die.
Consider equilibrium for this model.

Equilibrium for the predator-pray system

$$\begin{cases} -k(1-u)v - av^2 = 0, \\ (1-v)u - bu^2 = 0. \end{cases} \Rightarrow \begin{cases} -v(k - ku - av) = 0, \\ u(1 - v - bu) = 0. \end{cases}$$

```
>>> from sympy import *
>>> v,u,a,b,k,y1,y2=symbols('v,u,a,b,k,y1,y2')
>>> dv=k*(-1+u)*v-a*v*v
>>> du=(1-v)*u-b*u*u
>>> S=solve([k*(-1+u)*v-a*v*v,(1-v)*u-b*u*u],v,u)
>>>S
[(0, 0), (-k/a, 0), (0, 1/b), (-k*(b - 1)/(a*b + k), (a + k)/(a*b + k))]
```

Then equilibrium points are following

$$(0,0) \cup (0,1/b) \cup (-k/a,0) \cup \left(\frac{-(b-1)k}{k+ab}, \frac{k+a}{k+ab} \right).$$

Equilibrium for the predator-pray system

The linearized system at $(0,0)$: $v \sim y_1$, $u \sim y_2$:

```
>>>
(diff(du,v)*y1+diff(du,u)*y2).subs(v,S[0][0]).subs(u,S[0][1])
y2
>>>
(diff(dv,v)*y1+diff(dv,u)*y2).subs(v,S[0][0]).subs(u,S[0][1])
-k*y1
```

Hence:

$$\begin{cases} \dot{y}_1 = -ky_1, \\ \dot{y}_2 = y_2. \end{cases}$$

This system defines a saddle.

Equilibrium for the predator-pray system

The linearized system at $\left(-\frac{k}{a}, 0\right)$: $v + \frac{k}{a} \sim y_1$, $u \sim y_2$:

>>>
(diff(dv,v)*y1+diff(dv,u)*y2).subs(v,S[1][0]).subs(u,S[1][1])k*y1 - k**2*y2/a

>>>
(diff(du,v)*y1+diff(du,u)*y2).subs(v,S[1][0]).subs(u,S[1][1])y2*(1 + k/a)

$$\begin{cases} \dot{y}_1 = k y_1 - \frac{k^2}{a} y_2, \\ \dot{y}_2 = \left(1 + \frac{k}{a}\right) y_2. \end{cases}$$

Equilibrium for the predator-pray system

$$\begin{cases} \dot{y}_1 = ky_1 - \frac{k^2}{a} y_2, \\ \dot{y}_2 = \left(1 + \frac{k}{a}\right) y_2. \end{cases}$$

Eigenvalues of the matrix in the right-hand side are
$\lambda_1 = k,\ \lambda_2 = 1 + k/a$.
The singular point for this equation is an unstable knot.

Equilibrium for the predator-pray system

The linearized system at $(0, 1/b)$: $v \sim y_1$, $u - 1/b \sim y_2$:

```
>>>
(diff(dv,v)*y1+diff(dv,u)*y2).subs(v,S[2][0]).subs(u,S[2][1])k*y1*(-
1 + 1/b)
>>>
(diff(du,v)*y1+diff(du,u)*y2).subs(v,S[2][0]).subs(u,S[2][1])-
y2 - y1/b
```

Then:

$$\begin{cases} \dot{y_1} = \left(\frac{k}{b} - k\right) y_1, \\ \dot{y_2} = -\frac{1}{b} y_1 - y_2. \end{cases}$$

Equilibrium for the predator-pray system

$$\begin{cases} \dot{y}_1 = \left(\frac{k}{b} - k\right) y_1, \\ \dot{y}_2 = -\frac{1}{b} y_1 - y_2. \end{cases}$$

Eigenvalues of the matrix in th right-hand side are
$\lambda_1 = k(1/b - 1),\ \lambda_2 = -1.$
The singular point for this equation is a stable knot for $b > 1$ and a saddle in opposite case.

Equilibrium for the predator-pray system

The linearized system at $\left(\frac{-(b-1)k}{k+ab}, \frac{k+a}{k+ab}\right)$:

$v + \frac{(b-1)k}{k+ab} \sim y_1$, $u - \frac{k+a}{k+ab} \sim y_2$:

```
>>>(diff(dv,v)*y1+diff(dv,u)*y2).subs(v,S[3][0]).subs(u,S[3][1])
-k**2*y2*(b - 1)/(a*b + k) + y1*(2*a*k*(b - 1)/(a*b + k)
+ k*((a + k)/(a*b + k) - 1))
>>>
(diff(du,v)*y1+diff(du,u)*y2).subs(v,S[3][0]).subs(u,S[3][1])
-y1*(a + k)/(a*b + k) + y2*(-2*b*(a + k)/(a*b + k) +
k*(b - 1)/(a*b + k) + 1)
```

As a result we get:

$$\begin{cases} \dot{y}_1 = \frac{ak(b-1)}{ab+k} y_1 - \frac{k^2(b-1)}{ab+k} y_2, \\ \dot{y}_2 = \frac{k+a}{ab+k}(-y_1 - by_2). \end{cases}$$

Equilibrium for the predator-pray system

$$\begin{cases} \dot{y}_1 = \frac{ak(b-1)}{ab+k} y_1 - \frac{k^2(b-1)}{ab+k} y_2, \\ \dot{y}_2 = \frac{k+a}{ab+k}(-y_1 - by_2). \end{cases}$$

Let's consider for simplicity following values for the coefficients: $k = 1$, $b = 0$, $a = 0$:

$$\begin{cases} \dot{y}_1 = y_2, \\ \dot{y}_2 = -y_1. \end{cases}$$

Then the eigenvalues are $\lambda_1 = i$, $\lambda_2 = -i$ and the singular point for the given linear system is a center.

Autonomous equations

A system of equations which does not contain the independent variable is called *autonomous system*

$$
\begin{aligned}
y &= f(y),\\
\dot{y}_1 &= f_1(y_1, y_2),\\
\dot{y}_2 &= f_2(y_1, y_2).
\end{aligned}
$$

The equation for a pendulum is a typical example of the autonomous system:

$$
\ddot{\phi} + \sin(\phi) = 0 \Rightarrow \begin{cases} y_1' = & y_2, \\ y_2' = & -\sin(y_1). \end{cases}
$$

Autonomous systems

The predator-pray system is autonomous system:

$$\begin{cases} \dot{y}_1 = & y_1 - y_1 y_2, \\ \dot{y}_2 = & k y_2(-1 + y_1). \end{cases}$$

Any non-autonomous system can be rewritten as autonomous one:

$$\dot{\mathbf{y}} = f(\mathbf{y}, t), \text{ define } y_{n+1} = t \Rightarrow$$
$$\begin{cases} \dot{y}_k = f_k(y_1, \ldots, y_{n+1}), k = 1, \ldots, n; \\ \dot{y}_{n+1} = 1. \end{cases}$$

Properties of solutions of the autonomous systems

If $\mathbf{y}(t)$ is a solution of a system $\dot{\mathbf{y}} = f(\mathbf{y})$, then the vector $\mathbf{y}(t + \text{const})$ is also the solution of the same system:

$$\frac{d\mathbf{y}(t + \text{const})}{dt} = \frac{d\mathbf{y}(t + \text{const})}{d(t + \text{const})} = f(\mathbf{y}).$$

Let the solution $\mathbf{y}(t) \equiv (y_1(t), y_2(t), \ldots, y_n(t))$ be known on interval $t \in [a, b]$, then the set $(y_1(t), y_2(t), \ldots, y_n(t))$ defines a curve in the n-th dimensional space $\mathbb{R}^n$.
Such curve is called phase curve and the space where these curve is defined is called phase space.

A theorem about phase curves

Let given point a is not equilibrium. Then the system of equations

$$\dot{x}_1 = f_1,\ x_2 = f_2, \ldots, \dot{x}_n = f_n$$

can be locally transformed into the form:

$$\dot{y}_1 = 0, \dot{y}_2 = 0, \ldots, \dot{y}_n = 1.$$

The trajectories are straight lines:

$$y_j = c_j, j = 1, \ldots, n-1, \quad y_n = c_n + t,\ c_k = \text{const}.$$

A theorem about straitening of phase curves

Let $\mathbf{f}(\mathbf{a}) \neq 0$. Without loss of a generality assume that $f_n(\mathbf{a}) = b_n \neq 0$ and $\mathbf{x}(t) = \phi(\xi, t), \quad \mathbf{x}(0) = \xi$. Then $\mathbf{x}$ can be represented as a solution of the integral equation:

$$\mathbf{x} = \xi + \int_0^t \mathbf{f}(\mathbf{x}, \tau) d\tau.$$

define $y_1 = \xi_1,\ y_2 = \xi_2, \ldots, y_{n-1} = \xi_{n-1},\ y_n = t. \Rightarrow$

$$A = \frac{\partial(\mathbf{y})}{\partial(\mathbf{x})} = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & b_n \end{pmatrix}$$

Therefore the inverse transform $\mathbf{y} = \phi^{-1}(\mathbf{x}, t)$ exists. The theorem is proved.

An example

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1.$$

$$x_1 = \xi_1 \cos(t) + \xi_2 \sin(t),$$

$$x_2 = -\xi_1 \sin(t) + \xi_2 \cos(t).$$

$$y_1 = \sqrt{\xi_1^2 + \xi_2^2},\ y_2 = t.$$

Liouville's theorem about evolution of a phase volume

Let's consider a phase volume

$$d\mathbf{V} = dx_1 dx_2 \dots dx_n = \prod_{k=1}^{n} dx_k$$

and define the evolution with respect to the differential system:

$$\frac{d\mathbf{V}(t+dt) - d\mathbf{V}(t)}{dt} = \sum_{j=1}^{n} \frac{(d_j(x_j + f_j dt) - dx_j)}{dt} \prod_{k \neq j} dx_k =$$

$$\sum_{j=1}^{n} \frac{\partial f_j}{\partial x_j} dx_j \prod_{k \neq j} dx_k = \mathbf{div}\,\mathbf{f} \prod_{k=1}^{n} dx_k.$$

An example

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\mu x_2 - \sin(x_1).$$

$$\frac{dV}{dt} = \mathbf{div}(x_2, -\mu - \mu x_2 - \sin(x_1)) dx_1 dx_2$$

$$\frac{dV}{dt} = -\mu V.$$

Phase curves for the pendulum

Let's consider the sum of kinetic and potential energy of the pendulum:

$$E = \frac{\dot{\phi}^2}{2} - \cos(\phi),$$

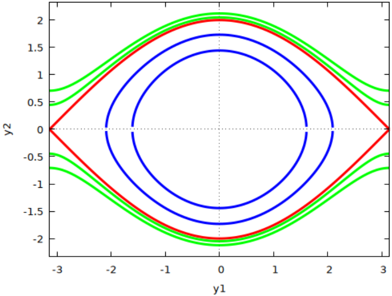
$$\frac{dE}{dt} = \dot{\phi}\ddot{\phi} + \sin(\phi)\dot{\phi} = \dot{\phi}(\ddot{\phi} + \sin(\phi)) = 0, \Rightarrow$$

The full energy is a conservation law for the pendulum. These property can be used for defining phase curves.

$$\frac{y_2^2}{2} - \cos(y_1) = E, \quad y_2 = \pm\sqrt{2E + 2\cos(y_1)},$$

$$y_2 \in \mathbb{R},\ y_1 \in \mathbb{S} \Rightarrow (y_1, y_2) \in \mathbb{S} \times \mathbb{R}.$$

Phase curves for the pendulum



- $E = -1,\ (y_1, y_2) = (0, 0)$;
- $-1 < E < 1,\ y_2 = \pm\sqrt{2E + 2\cos(y_1)},\ y_1 \in [-\arccos(E), \arccos(E)]$ (blue curves);
- $E = 1,\ (y_1, y_2) \in (0, \pi) \cup y_2 = \pm\sqrt{2E - 2\cos(y_1)}, y_1 \in (-\pi, \pi)$(red curve);
- $1 < E,\ y_2 = \pm\sqrt{2E - 2\cos(y_1)},\ y_1 \in [-\pi, \pi)$ (green curves).

The conservation law for the predator-prey model

Let us divide the equation

$$\frac{dv}{d\tau} = -k(1-u)v,$$

by the equation

$$\frac{du}{d\tau} = (1-v)u.$$

As a result we obtain:

$$\frac{dv}{du} = \frac{-k(1-u)v}{(1-v)u}.$$

The conservation law for the predator-prey model

Then rewrite the equation in the differential form:

$$(1-v)\frac{dv}{v} = -k(1-u)\frac{du}{u}$$

or

$$\frac{dv}{v} - dv = kdu - k\frac{du}{u}.$$

After integrating we get:

$$\log(v) - v = -k\log(u) + ku + C.$$

The conservation law for the predator-prey model

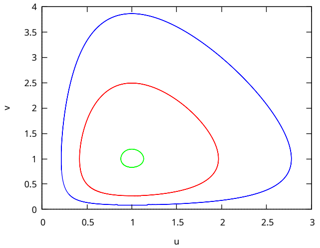


Figure: The phase portrait of the predator-prey model, $k = 2$.

The value

$$C = \log(vu^k) - (ku + v)$$

is a conservation law for the predator-prey model:

$$\frac{dC}{d\tau} = \frac{dv}{d\tau}\frac{u^k}{vu^k} + k\frac{du}{d\tau}\frac{u^{k-1}v}{vu^k} - k\frac{du}{d\tau} - \frac{dv}{d\tau} = -k(1-u) + k(1-v) - k(1-v)u + k(1-u)v = -k + ku + k - kv - ku + kvu + kv - kuv = 0.$$

Conservation law

The function $U(\mathbf{x})$ is a conservation law of the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

if

$$\sum_{k=1}^{n} \frac{\partial U}{\partial x_k} f_k(\mathbf{x}) = 0.$$

Non-conservative pendulum

Let's consider a pendulum with friction:

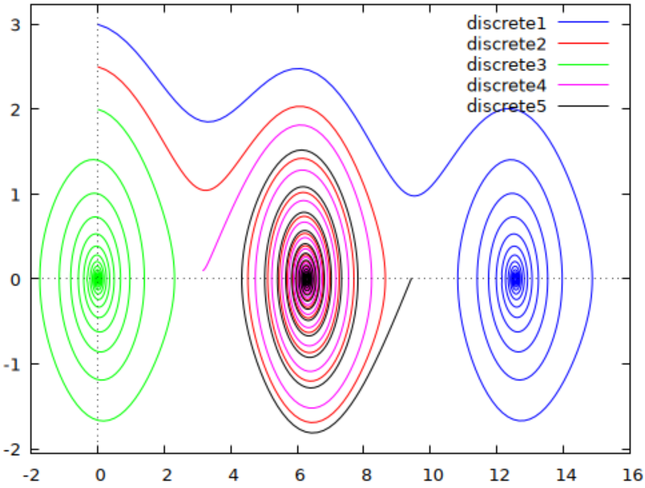
$$\ddot{\phi} + \mu\dot{\phi} + \sin(\phi) = 0 \Rightarrow \begin{cases} y_1' = & y_2, \\ y_2' = & -\mu y_2 - \sin(y_1). \end{cases}$$

Here $\mu > 0$ is a friction coefficient.
Find evolution of the full energy:

$$\frac{dE}{dt} = \dot{\phi}\ddot{\phi} + \sin(\phi)\dot{\phi} = \dot{\phi}(\ddot{\phi} + \sin(\phi)) = -\mu\dot{\phi}^2.$$

The energy of the pendulum with friction decreases.

Non-conservative pendulum



A predator pray system with competing species

$$\frac{dv}{d\tau} = -k(1-u)v - av^2, \frac{du}{d\tau} = (1-v)u - bu^2.$$

The derivative of the conservation law for the predator-pray system:

$$C = \log(vu^k) - (ku + v),$$
$$\frac{dC}{d\tau} = \left(\frac{1}{v} - 1\right)\dot{v} + k\left(\frac{1}{v} - 1\right)\dot{u} - k\dot{u} - \dot{v} =$$
$$a(v^2 - v) + bk(u^2 - u).$$

As a result one gets that the C changes under evolution of the system with competing species.

First definition of the derivative

A derivative is a limit:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{y(x + \Delta x) - y(x)}{\Delta x}.$$

$\forall \varepsilon > 0,\ \exists \delta(\varepsilon) > 0,\ |\Delta x| < \delta$:

$$\left| \frac{y(x + \Delta x) - y(x)}{\Delta x} - \frac{dy}{dx} \right| < \varepsilon.$$

This definition appeals to properties of the curve for a continuous parameter Δx. While a numeric approach assumes a discrete set of allowed values of Δx.

Second definition of the derivative

$$\frac{dy}{dx} = \lim_{n\to\infty} \frac{y(x+\Delta x_n) - y(x)}{\Delta x_n}.$$

$\forall \varepsilon > 0$ and any $\{\Delta x_n\}_{n=1}^{\infty} \to 0,\ \exists N(\varepsilon)$ and:

$$\left| \frac{y(x+\Delta x_n) - y(x)}{\Delta x_n} - \frac{dy}{dx} \right| < \varepsilon.$$

This concept looks close to the numeric approach but basic details are the words any and convergent sequence.

Disclaimer

When we deal with numeric formulas for the derivative then we must understand that we neglect two important ideas of the classical mathematical analysis:

- continuous function;
- continuous of independent variable.

We change the continuous by the a set of values of the floating point numbers.

Disclaimer

Despite of following examples we will assume that the numbers are a set on a lattice with uniform step Δ..

- Noncommutativity:

```
> > >2.0-0.3+0.3==2.0
true
> > >2.0+0.3-0.3==2.0
false
```

- An errors due to averaging by standard IEEE754

```
> > >z=2.0-0.9
> > >"%.18f" % z
'1.100000000000000089'
```

Approximations of derivatives

Let's consider a Taylor formula for a smooth function $f(x)$ with a residue in the Lagrange form:

$$f(x+\Delta) = f(x) + f'(x)\Delta + O(\Delta^2).$$

It yields:

$$f'(x) = \frac{f(x+\Delta) - f(x)}{\Delta} + O(\Delta).$$

Two-point formula for the derivative

If we consider three terms of the Taylor series:

$$f(x+\Delta) = f(x) + f'(x)\Delta + \frac{f''(x)}{2!}\Delta^2 + O(\Delta^3).$$

Here second derivative is:

$$f''(x) = \frac{1}{\Delta}\left(\frac{f(x+\Delta)-f(x)}{\Delta} - \frac{f(x)-f(x-\Delta)}{\Delta}\right) + O(\Delta) =$$
$$\frac{f(x+\Delta)-2f(x)+f(x-\Delta)}{\Delta^2} + O(\Delta).$$

In this case:

$$f(x+\Delta) = f(x) + f'(x)\Delta + \frac{f(x+\Delta)-2f(x)+f(x-\Delta)}{2!} + O(\Delta^3)$$

and

$$f'(x) = \frac{f(x+\Delta)-f(x-\Delta)}{2\Delta} + O(\Delta^2).$$

Two two-point formulas for the derivative

$$
\begin{aligned}
f'(x) &= \frac{f(x+\Delta) - f(x)}{\Delta} + O(\Delta), \\
f'(x) &= \frac{f(x+\Delta) - f(x-\Delta)}{2\Delta} + O(\Delta^2).
\end{aligned}
$$

Four-point formulas for the derivative

The four terms of the Taylor series:

$$f(x+\Delta) \sim f(x) + f'(x)\Delta + \frac{f''(x)}{2!}\Delta^2 + \frac{f'''(x)}{3!}\Delta^3.$$

Here

$$\begin{aligned} f'''(x) &= \frac{f''(x+\Delta) - f''(x-\Delta)}{2\Delta} + O(\Delta^2). \\ f'''(x) &= \frac{1}{2\Delta^3}\left(f(x+2\Delta) - 2f(x+\Delta) + f(x)\right) - \\ &- \frac{1}{2\Delta^3}\left(f(x) - 2f(x-\Delta) + f(x-2\Delta)\right) \end{aligned}$$

Then:

$$f'(x) = \frac{1}{12\Delta}(f(x+2\Delta) - 8f(x+\Delta) + 8f(x-\Delta) - f(x-2\Delta)) + O(\Delta^4).$$

Formulas for first derivative

$$
\begin{aligned}
f'(x) &= \frac{f(x+\Delta)-f(x)}{\Delta} + O(\Delta), \\
f'(x) &= \frac{f(x+\Delta)-f(x-\Delta)}{2\Delta} + O(\Delta^2), \\
f'(x) &= \frac{1}{12\Delta}(f(x+2\Delta) - 8f(x+\Delta) + \\
&\quad +8f(x-\Delta) - f(x-2\Delta)) + O(\Delta^4).
\end{aligned}
$$

Examples

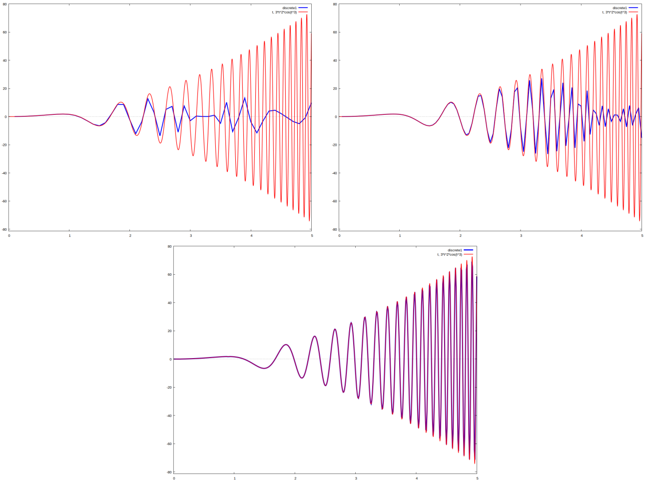


Figure: The numeric derivative of $\sin(x^3)$ on the interval $x \in [0, 5]$ with steps 0.1, 0.05, 0.01.

The Euler-Cauchy method for $y' = f(x, y)$

Let's consider the three terms of the Taylor series:

$$y(x+\Delta) = y(x) + f(x, y(x))\Delta + \frac{1}{2}\frac{d^2y}{dx^2}\Delta^2 + O(\Delta^3).$$

Let's define:

$$x_{n+1} = x_n + \Delta, \quad y(x_{n+1}) = y_{n+1};$$

$$y'' = \frac{1}{\Delta}(f(x_{n+1}, y_n + f(x_k, y_k)\Delta) - f(x_n.y_n)) + O(\Delta).$$

Then

$$y_{n+1} = y_n + f(x_n, y_n)\Delta +$$

$$\frac{1}{2\Delta}\left(f(x_{n+1}, y_n + f(x_n, y_n)\Delta) - f(x_n, y_n)\right)\Delta^2 + O(\Delta^3),$$

$$y_{n+1} = y_n + (f(x_n, y_n) + f(x_{n+1}, y_n + f(x_n, y_n)\Delta))\frac{\Delta}{2} + O(\Delta^3).$$

The Euler-Cauchy method for $y' = f(x, y)$

Denote:

$$k_1 = f(x_n, y_n), \quad k_2 = f(x_{n+1}, y_n + k_1\Delta),$$

$$y_{n+1} = y_n + \frac{\Delta}{2}(k_1 + k_2) + O(\Delta^3).$$

Taylor series and accuracy

$$y_{n+1} = y_n + f(x_n, y_n)\Delta + \frac{\Delta^2}{2}\frac{df}{dx} + \frac{\Delta^3}{3!}\frac{d^2 f}{dx^2} + O(\Delta^4).$$

Here

$$\frac{df}{dx} = \frac{d}{dx}(f(x, y(x)) = f'_x + f'_y f,$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}(f'_x + f'_y f) = f''_{xx} + 2f''_{xy} f + f''_{yy} f^2 + f'_y f'_x + (f'_y)^2 f.$$

For the calculations these formulas are rare used.

A formulas for the Runge-Kutta of fourth-order

$$y_{n+1} = y_n + \frac{\Delta}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

$$k_1 = f(x_n, y_n), \quad k_2 = f\left(x_n + \frac{\Delta}{2}, y_n + k_1\frac{\Delta}{2}\right)$$

$$k_3 = f\left(x_n + \frac{\Delta}{2}, y_n + k_2\frac{\Delta}{2}\right) \quad k_4 = f\left(x_n + \Delta, y_n + k_3\Delta\right)$$

An example

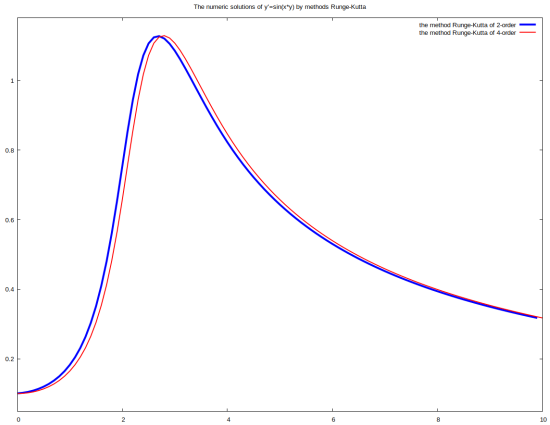


Figure: Here one can see results of numerical calculations for the equation $y' = \sin(xy)$, $y|_{x=0} = 0.1$ by the second and fourth order method Runge-Kutta with step 0.1.

Software for numeric calculations

- Computer algebra systems: Maxima (free software), Mathematica, Maple.
- Software for numeric simulations: Scilab (free and open-source), Mathlab.
- library for the scientific calculations: scipy.