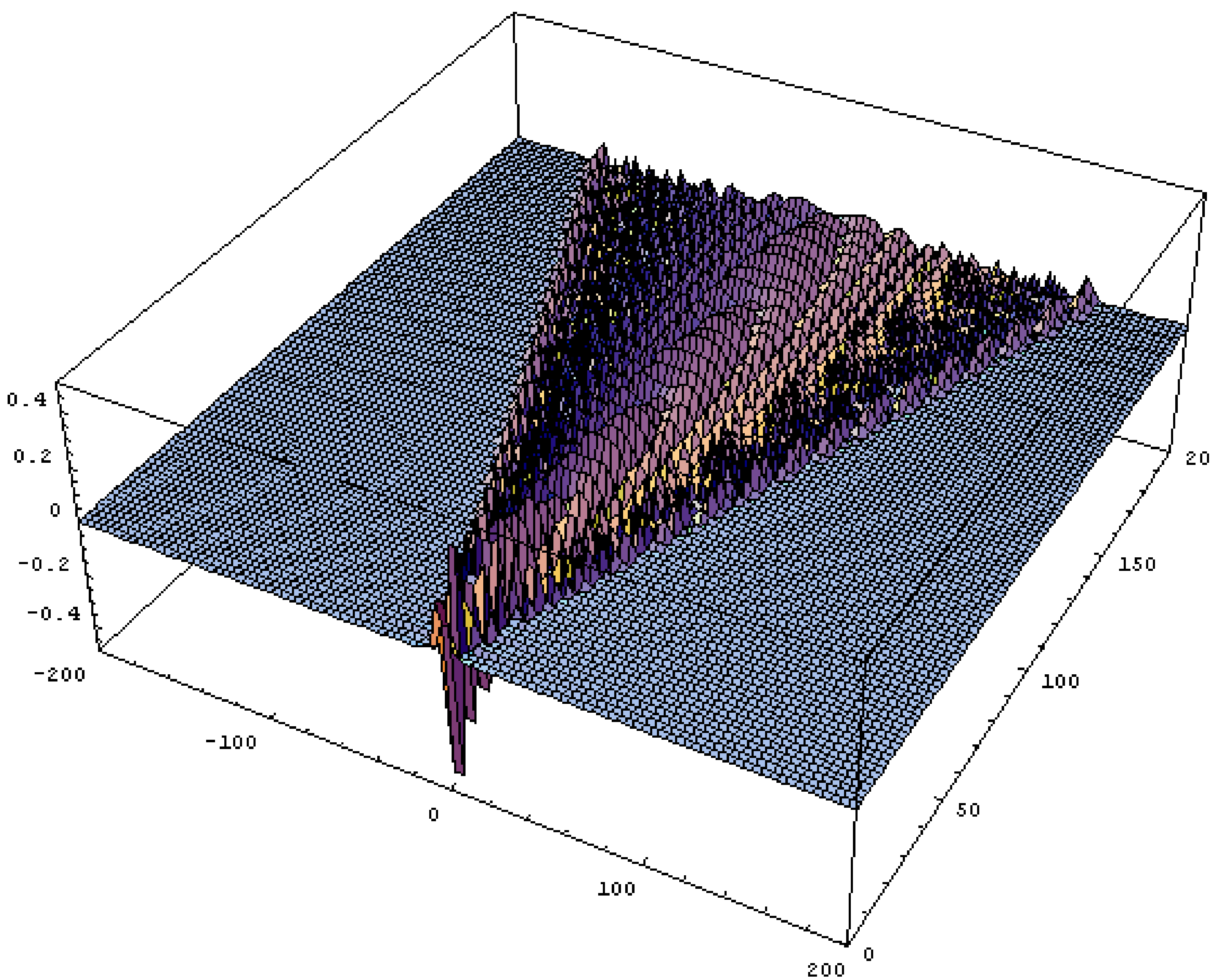
Soliton generation by local resonance interaction

S.G. Glebov,
Ufa State Petroleum
Technical University,
E-mail: sg@anrb.ru

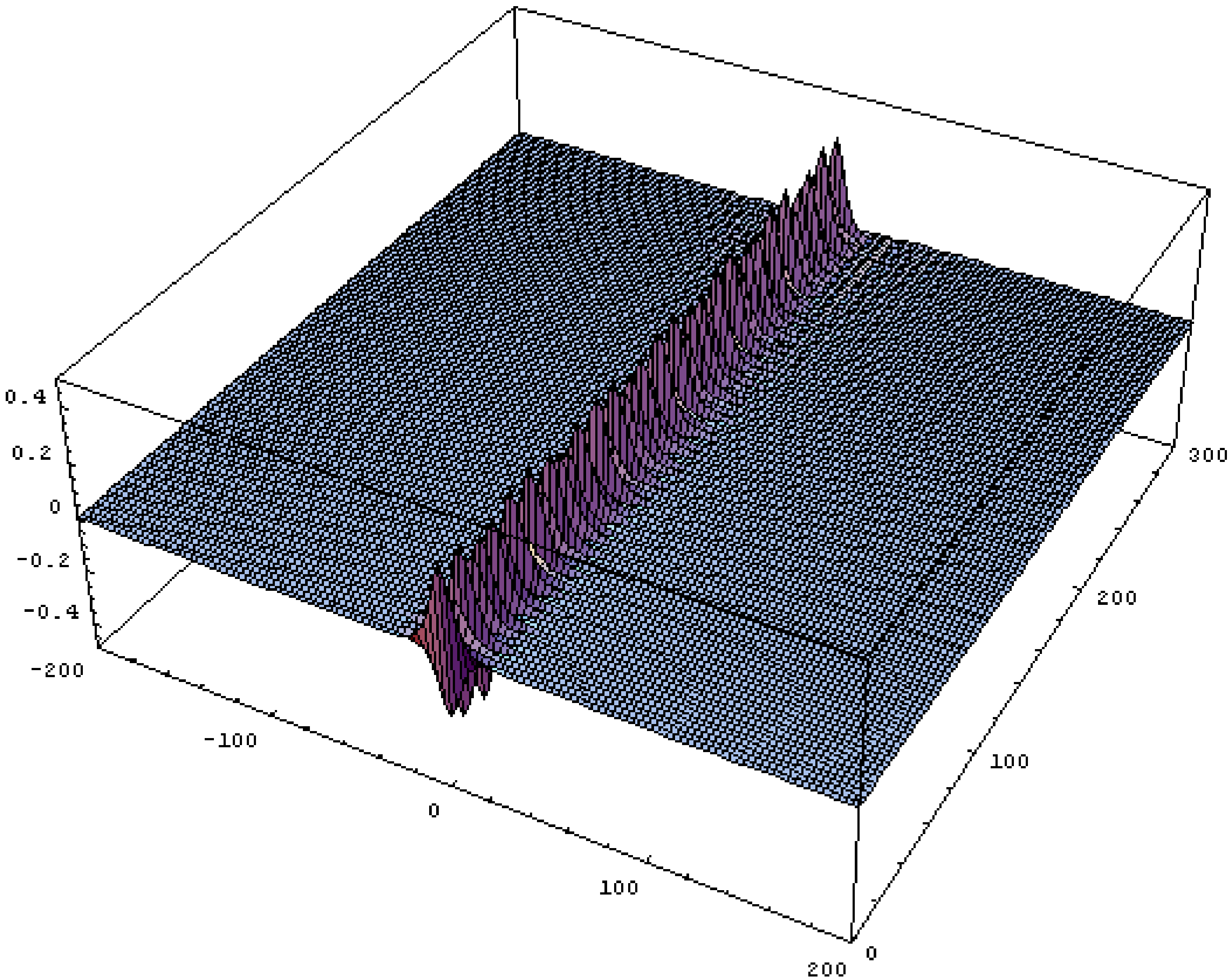
O.M. Kiselev,
Institute of Mathematics
USC of RAS,
E-mail: ok@ufanet.ru

V.A. Lazarev,
Ufa State Petroleum
Technical University,
E-mail: lazva@mail.ru

The propagation of the optical waves in fibers without of distortion for the envelope is one of main problems for the nonlinear optics. The distortion of the envelope appears due to the dispersion, nonlinearity and dissipation. The dissipation control is a separate topic and we do not discussed here.



The dispersion and nonlinearity are opposed by each other. The dispersion leads to go to the pieces of the packet and the nonlinearity tends to gather the packet. In a special case there exists a magic relation between the typical scales of the packets such as the amplitude and length of the wave in the packets. In this case the envelope of the wave packet is solution of the Nonlinear Shr'́odinger equation. There was founded by Kelley, Talanov and Zakharov in 1964-65 years. Later the nonlinear Shródinger equation was integrated by inverse scattering transform (Zakharov and Shabat 1971). The Nonlinear Schr'́odinger equation have a special solutions called by solitons which propagate without the distortion of the shape.



The solitary packets of waves would be more suitable for communication in optical fibers on a large distance if one can control the parameters of the envelope function for such packets.

It is well known two ways for making a solitary packets of waves. One of them is a spontaneous generation from an initial profile of the wave pocket. Such method for the soliton generation used the asymptotic behaviour of the soliton equations such as the Nonlinear Shr'odinger equation. The solitonic envelope is formed as an asymptoic limit for the long time. This fact was opened by Manakov, Ablowitz very soon as the NLSE was integrated in 1973.

Another one way is using the transverse instability of the waves in the nonlinear medium. The instability was founded by Kadomtsev and Petviashvily for the waves in plasma (1973). The same mechanism of the instability is used for the making the solitonic profile in the nonlinear equations.

One of the main difficulty for the making solitonic profiles for the envelope of wave packets is controlling of the parameters for the solitons. Both of the ways for the making of solitons as an envelope of the wave packets using the initial data for obtaining of the needed result. It is known that the solitons are unstable with respect to the initial data. The parameters of the solitons defined the position and the velocity soliton. So to make the soliton with fixed parameters one need to solve the instable problem with respect to the initial data.

It is easy to see the parameters of such self-generated solitons are predictable with difficulty in practice. It is explained by an instability of the parameters for solitons with respect to initial data.

We demonstrate a new approach for the making of the solitary packet of waves with solitonic envelope. The main idea is to use the small perturbation to control for the process of making the solitons.

In our approach the wave packets appear due to a slowly passage of the external driving force through the resonance. After the resonance the envelope function of the wave packet is determined by the nonlinear Schrödinger equation (NLSE). In the most important cases the envelope function is a sequence of solitary waves which are called solitons. The wave packets with the solitons as the envelope function are propagated without a dissipation. The parameters of the solitons are obviously defined by the value of the driving force on a resonance curve. We demonstrate this phenomenon for the perturbed nonlinear Klein-Gordon equation.

Statement of the problem

Let us consider the Klein-Gordon equation with a cubic nonlinearity

$$\partial_t^2 U - \partial_x^2 U + U + \gamma U^3 = \varepsilon^2 f(\varepsilon x) \exp\left\{ i\frac{S(\varepsilon^2 t, \varepsilon^2 x)}{\varepsilon^2} \right\} + \text{c.c.}. \tag{1}$$

Here $0 < \varepsilon \ll 1$, $\gamma = \text{const}$; $f(y)$ is smooth and rapidly vanishes as $y \to \pm\infty$. The function $S(y, z)$ and all derivatives with respect to y, z are bounded. Here and below we use the following notations

$$x_j = \varepsilon^j x, \quad t_j = \varepsilon^j t, \quad j = 1, 2;$$

$$l(t_2, x_2) \equiv (\partial_{t_2} S)^2 - (\partial_{x_2} S)^2 - 1.$$

We will construct a special asymptotic solution of equation (1) such that:

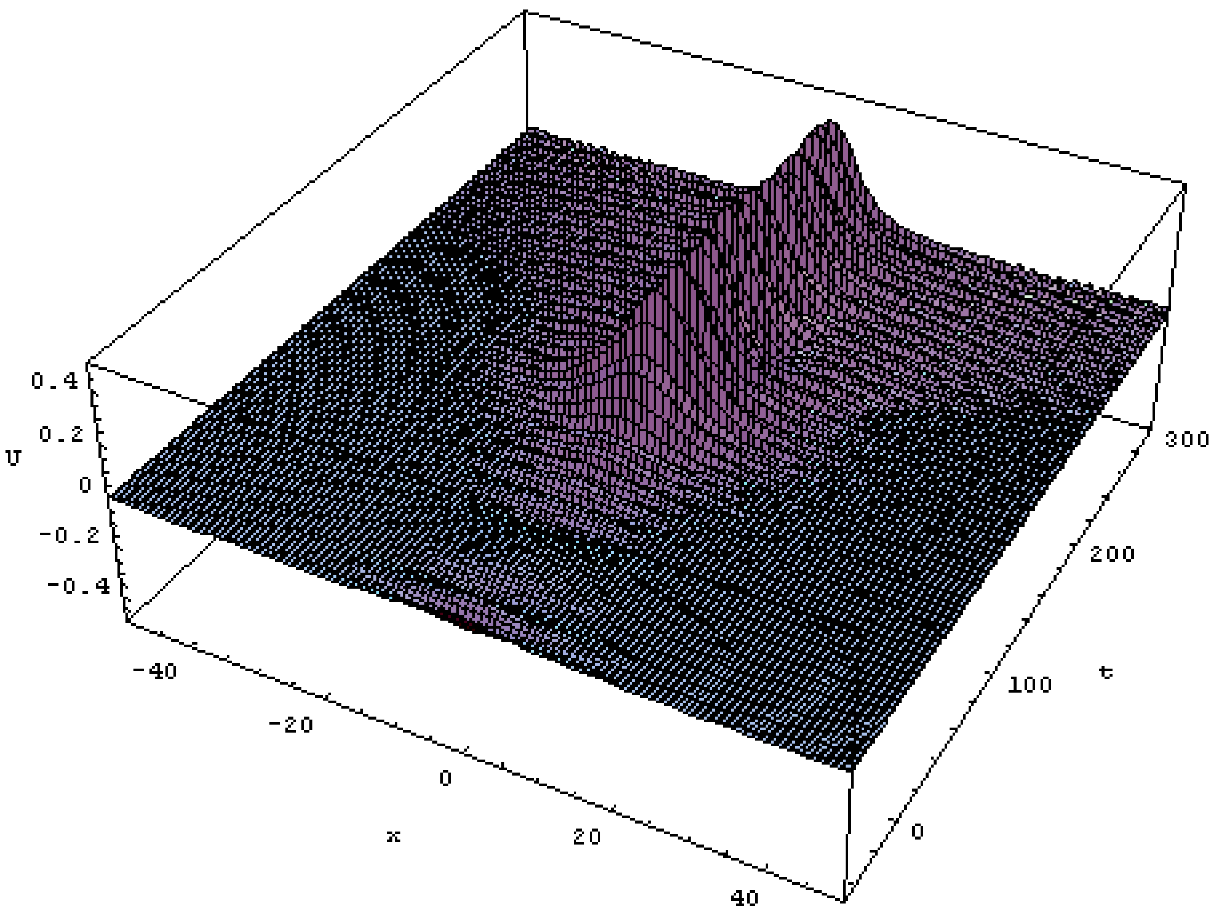
$$U \sim -\varepsilon^2 \frac{f}{l} \exp(iS(t_2, x_2)/\varepsilon^2) + c.c.. \tag{2}$$

when $l < 0$.

Numeric simulations

To illustrate the obtained result we consider equation (1) with $\gamma = 2$ and the simplest driving force, where

$$S = \frac{t_2^2}{2}, \quad f = \frac{2\sqrt{2}}{\sqrt{\pi}\cosh(2x_1)}. \tag{3}$$

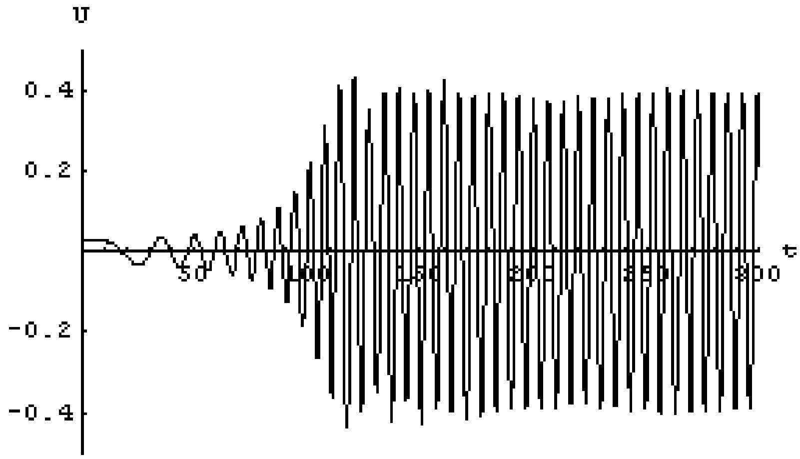


This picture shows the generation of the solitary packet of waves for equation (1) with special right-hand side(3), and at $\varepsilon = 0.1$. Initial conditions are

$$U|_{t=0} = -\varepsilon^2 f \exp(iS/\varepsilon^2)|_{t=0},$$

$$\partial_t U|_{t=0} = -\varepsilon^2 \partial_t (f \exp(iS/\varepsilon^2))|_{t=0}.$$

The resonant curve is $t = 100$.



This picture shows a profile $(U(x,t)|_{x=0})$of the packet.

In this case the curve of the local resonance is the line $t_2 = 1$. The pre-resonant solution has the form:

$$U \sim \frac{-\varepsilon^2}{(t_2-1)}\frac{2\sqrt{2}}{\sqrt{\pi}\cosh(2x_1)}\cos(it_2^2/\varepsilon^2), \quad 0 < t_2 < 1.$$

In the domain $t_2 > 1$ the solution has the form:

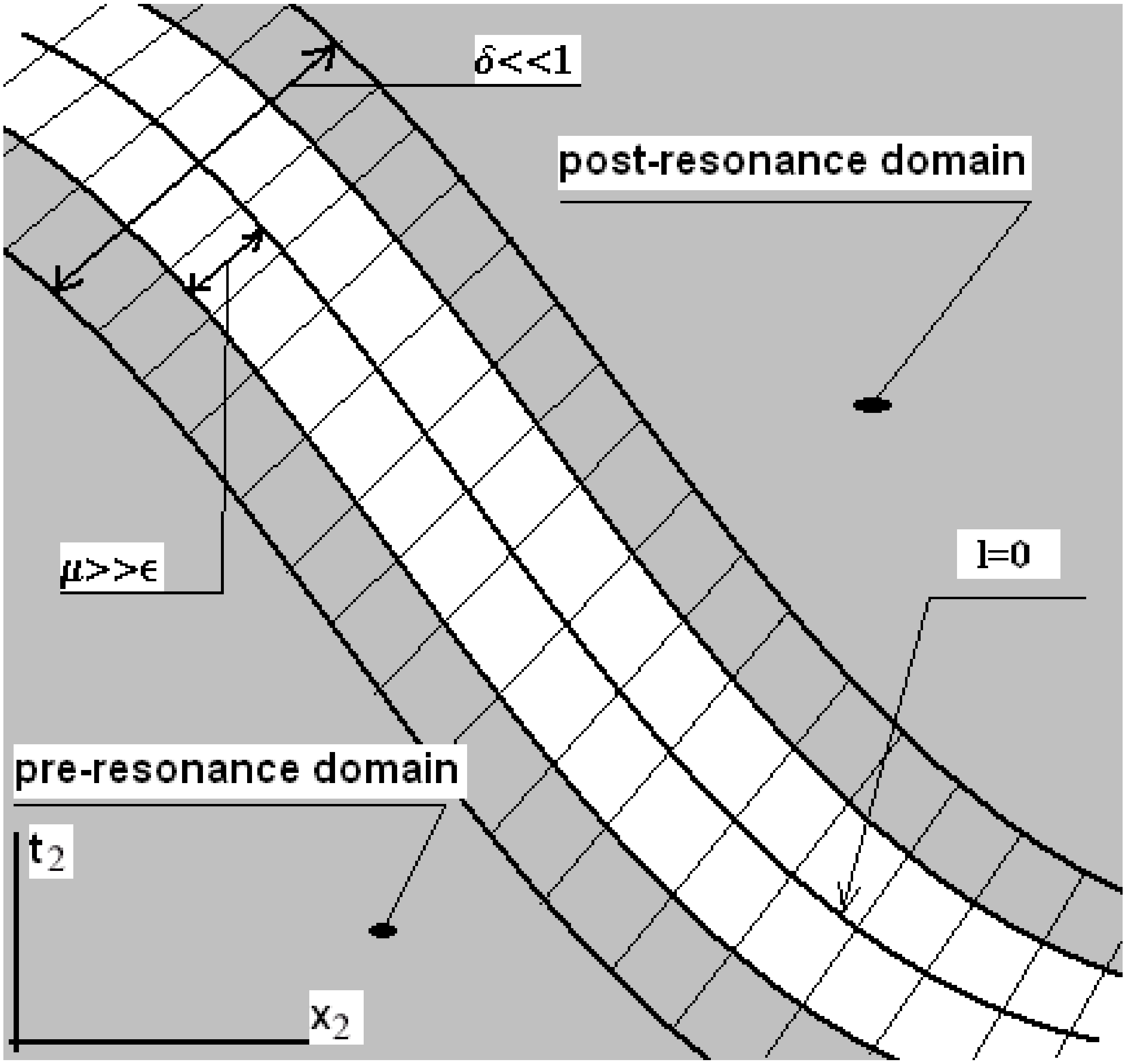
$$U \sim \varepsilon \exp\{i\varphi(x_2,t_2)/\varepsilon^2\}\Psi(x_1,t_1,t_2) + c.c.$$

Here $\varphi = t_2 - 1/2$. The function $\Psi(x_1,t_1,t_2)$ is the solution of the Cauchy problem for the NLSE:

$$2i\partial_{t_2}\Psi + \partial^2_{x_1x_1}\Psi + 2|\Psi|^2\Psi = 0,$$

$$\Psi|_{t_2=1} = \frac{\sqrt{2}(1+i)}{\cosh(2x_1)}.$$

Asymptotic analysis



All domains where we construct the solution is separated on three pairwise joint domains. The pre-resonant domain corresponds the forced oscillations with the amplitude of the order ε^2. This oscillations break down when the driving force becomes resonant. The resonant layer is a thin domain near the resonant curve $l(x_2, t_2) = 0$. In this layer the amplitude of the oscillations increases up to the order ε. In the post-resonant domain the amplitude of the solution stabilizes on the order of ε.

Pre-resonant expansion

In the domain $-l \gg \varepsilon$ the formal asymptotic solution of equation (1) modulo $O(\varepsilon^{N+1})$ has the form

$$U = \sum_{n\geq 2}^{N} \varepsilon^n U_n(t, x, \varepsilon), \quad (4)$$

where

$$U_n = \sum_{k\in\Omega_n} U_{n,k}(t_2, x_2, \varepsilon x) \exp\left\{ik\frac{S(t_2, x_2)}{\varepsilon^2}\right\}.$$

The set Ω_n for the higher-order term is described by the formula

$$\Omega_n = \begin{cases} \{\pm 1\}, & n \leq 5; \\ \{\pm 1, \pm 3, \dots, \pm(2l+3)\}, & l = \Big[(n-6)/4\Big], \quad n \geq 6. \end{cases}$$

The functions $U_{n,k}$ and $U_{n,-k}$ are complex conjugated.

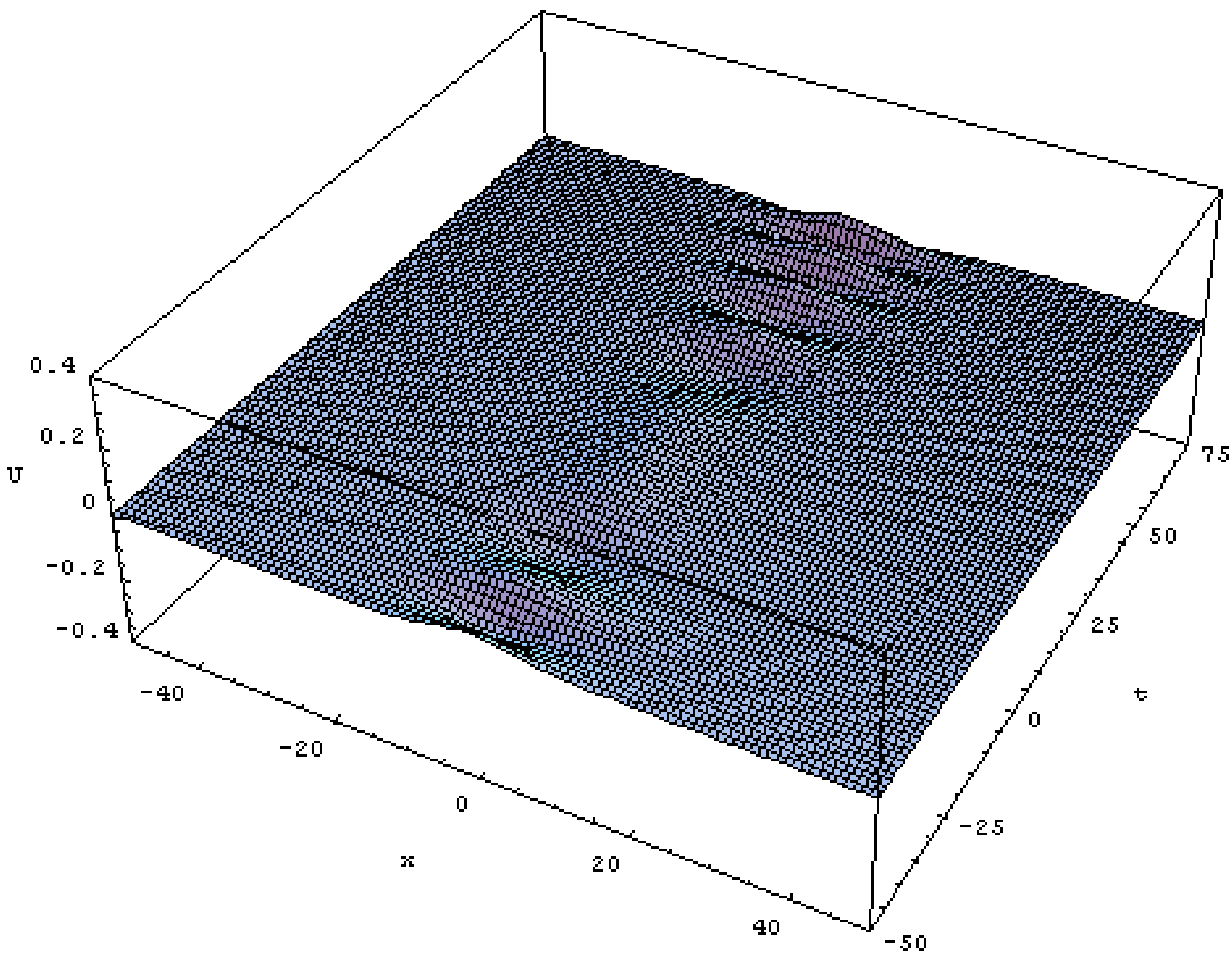
The coefficients of the asymptotics $U_{n,k}$ are defined out of algebraic equations

$$U_{2,1} = -\frac{f}{l}, \quad (5)$$

$$U_{3,1} = 2i\frac{\partial_{x_1} f \partial_{x_2} S}{l^2}, \quad (6)$$

$$U_{4,1} = \frac{2if[\partial_{t_2} S \partial_{t_2} l - \partial_{x_2} S \partial_{x_2} l] - 4(\partial_{x_2} S)^2 \partial_{x_1}^2 f}{l^3} - \frac{2i\partial_{t_2} f \partial_{t_2} S + \partial_{x_1}^2 f + i\partial_{t_2}^2 S f}{l^2}, \quad (7)$$

In this section we obtain the WKB-type of the asymptotic expansion which is valid before the resonance layer. This piece of the solution one can see on the following picture:



Resonant expansion

This part contains the asymptotic construction of the solution for equation (1) in the neighborhood of the curve $l = 0$. The domain of validity of this asymptotics intersects with the domain of validity of expansion (4). These expansions are matched.

In the domain $|l| \ll 1$ the formal asymptotic solution for equation (1) modulo $O(\varepsilon^{N+1})$ has the form

$$U = \sum_{n\geq 1}^{N} \varepsilon^n W_n(t_1, x_1, t_2, x_2, \varepsilon), \tag{8}$$

where

$$W_n = \sum_{k\in\Omega_n} W_{n,k}(x_2, t_2, x_1, t_1) \exp\left\{ik\frac{S(t_2, x_2)}{\varepsilon^2}\right\}, \tag{9}$$

The function $W_{n,1}$ is a solution of the problem for differential equations like the equation for the coefficient $W_{1,1}(x_1, t_1, x_2, t_2)$, which is defined by first order partial differential equation:

$$2i\partial_{t_2}S\partial_{t_1}W_{1,1} - 2i\partial_{x_2}S\partial_{x_1}W_{1,1} - \lambda W_{1,1} = f,$$

with a given asymptotic behaviour:

$$W_{1,1} \sim \frac{-f}{\lambda}, \quad \lambda \to -\infty.$$

Here $\lambda = l/\varepsilon$.

The asymptotic behaviour of $W_{1,1}$ as $\lambda \to \infty$ allows to relate the formulas (2) and (12).

The equation for $W_{1,1}$ may be written in the form of first order ordinary differential equation along the characteristic direction:

$$\frac{d}{d\sigma}W_{1,1} + \lambda W_{1,1} = f.$$

Such ordinary equation appears under studying of slowly passage through resonance for a one-dimensional oscillator with slowly varying frequency by Kevorkyan. The solution of equations of such type defines by Fresnel integrals.

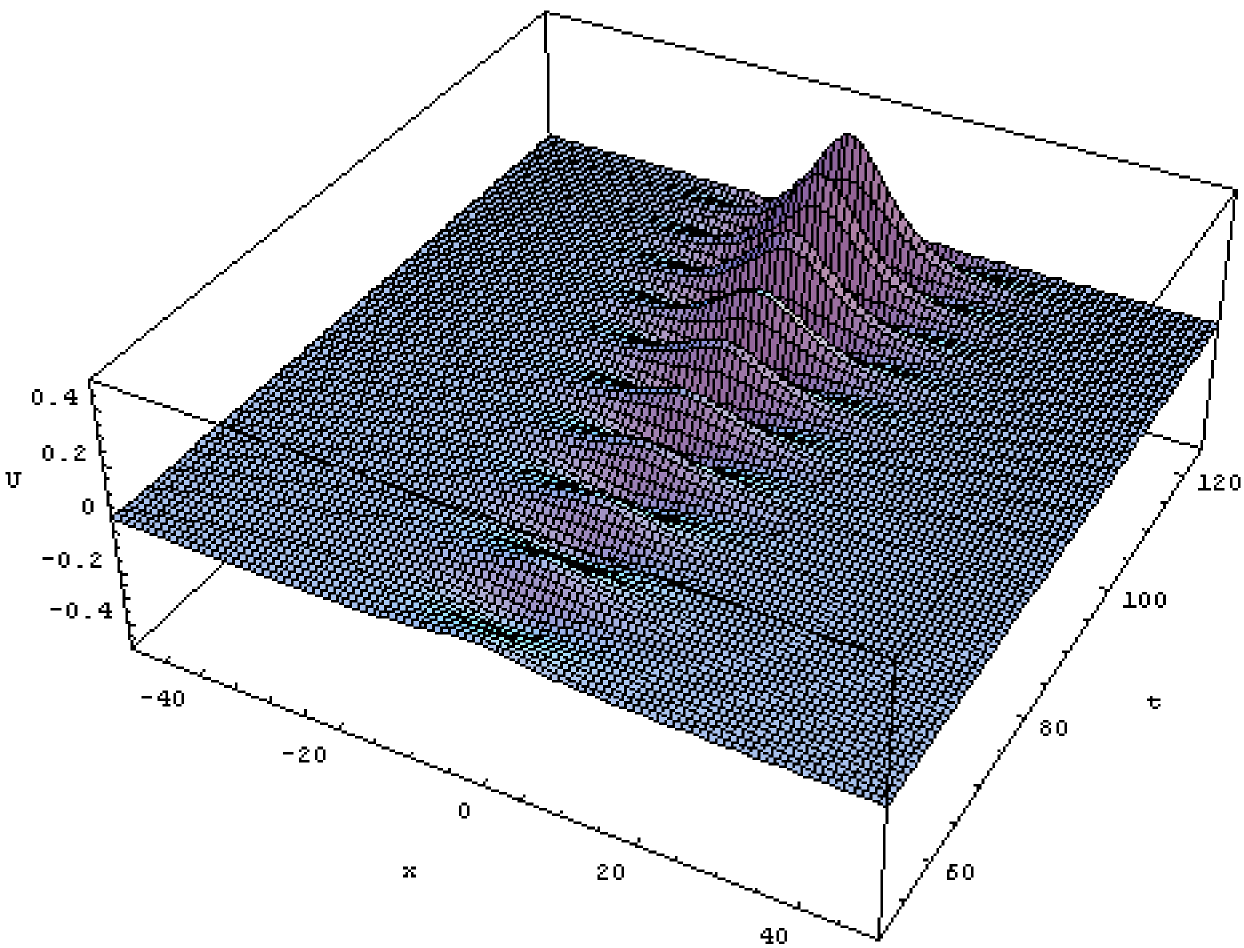
When $k \neq 1$ $W_{n,k}$ is the solution of algebraic equation. The functions $W_{n,k}$ and $W_{n,-k}$ are complex conjugated.

We obtain:

$$U(x,t,\varepsilon) \sim \varepsilon W_{1,1}(x_1,t_1,x_2,t_2)\,\mathrm{exp}\{iS/\varepsilon^2\} + c.c..$$

There is an essential difference between asymptotics (8) and external pre-resonance asymptotics (4). In the first place the leading-order term in (8) has an order ε while the leading order term in (4) has an order ε^2. In the second place the coefficients of asymptotics (8) depend on fast variables $x_1 = x_2/\varepsilon$ and $t_1 = t_2/\varepsilon$.

The resonant layer contains the strip where the solution increases due to the local resonance. This piece of the strip is shown on the following figure:



Post-resonant expansion

In the domain $l \gg \varepsilon$ the formal asymptotic solution of equation (1) modulo $O(\varepsilon^{N+1})$ has a form

$$U(x,t,\varepsilon) = \sum_{1}^{N} \varepsilon^n \sum_{k=0}^{n-2} \ln^k(\varepsilon) \times$$
$$\times \left(\sum_{\pm\varphi} \exp\{\pm i\varphi(x_2,t_2)/\varepsilon^2\} \Psi_{n,k,\pm\varphi}(x_1,t_1,t_2) + \right.$$
$$\left. \sum_{\chi \in K'_{n,k}} \exp\{i\chi(x_2,t_2)/\varepsilon^2\} \Psi_{n,k,\chi}(x_1,t_1,t_2) \right). \quad (10)$$

Here the function $\varphi(x_2,t_2)$ satisfies the eikonal equation

$$(\partial_{t_2}\varphi)^2 - (\partial_{x_2}\varphi)^2 - 1 = 0 \quad (11)$$

and initial condition on the curve $l = 0$:

$$\varphi|_{l=0} = S|_{l=0}, \quad \partial_{t_2}\varphi_{l=0} = \partial_{t_2} S|_{l=0}.$$

The leading-order term of the asymptotics is a solution of the Cauchy problem for the nonlinear Schrodinger equation

$$2i\partial_{t_2}\varphi\partial_{t_2}\Psi_{1,0,\varphi} + \partial_\xi^2\Psi_{1,0,\varphi} + i[\partial_{t_2}^2\varphi - \partial_{x_2}^2\varphi]\Psi_{1,0,\varphi} +$$
$$\gamma|\Psi_{1,0,\varphi}|^2\Psi_{1,0,\varphi} = 0,$$

$$\Psi_{1,0,\varphi}|_{l=0} = \int_{-\infty}^{\infty} d\sigma f(x_1) \exp(i\int_0^\sigma d\chi \lambda(x_1,t_1,\varepsilon)),$$

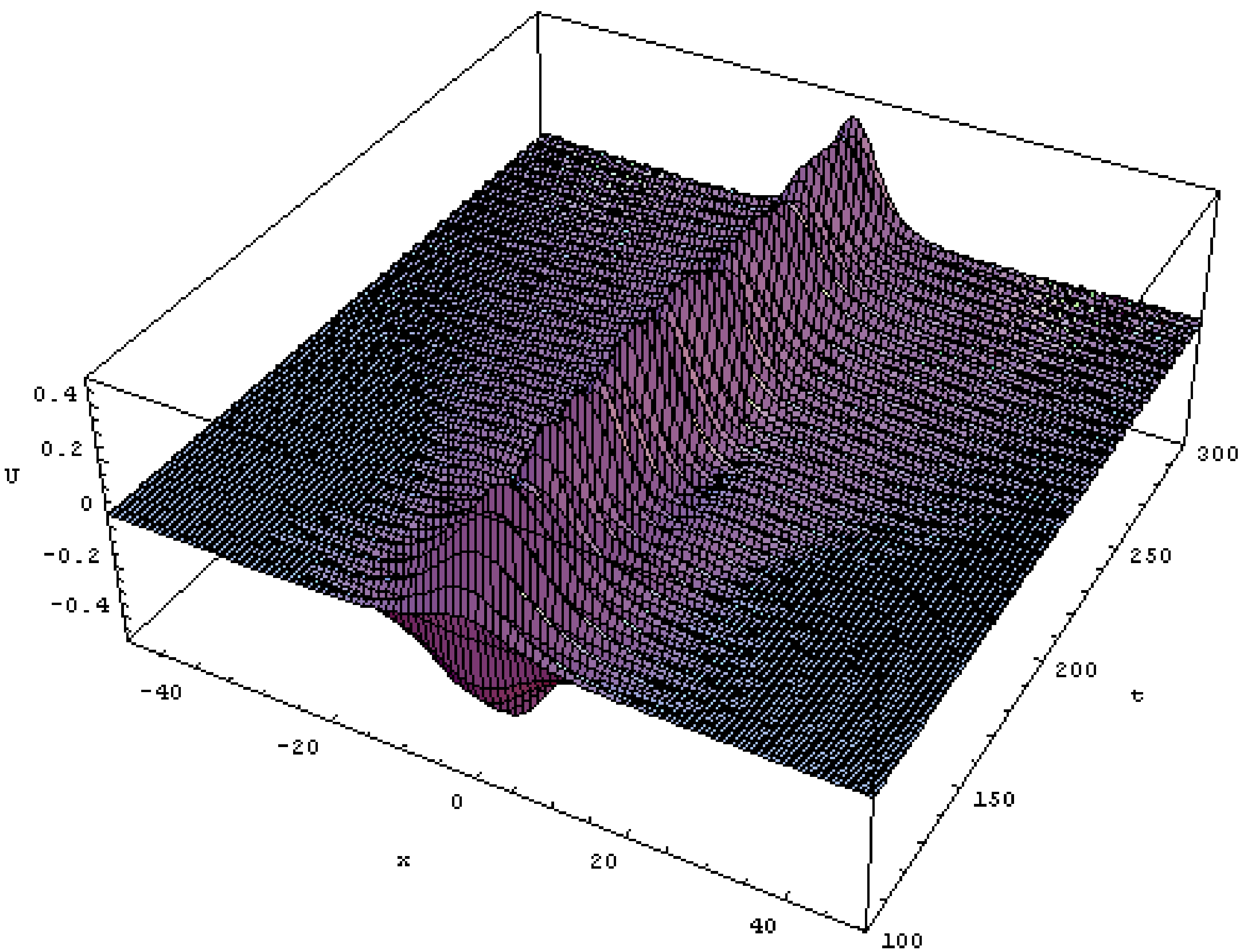
where ξ is defined by

$$\frac{dx_1}{d\xi} = \partial_{t_2}\varphi, \quad \frac{dt_1}{d\xi} = \partial_{x_2}\varphi.$$

The coefficients $\Psi_{n,k,\pm\varphi}$ are determined from Cauchy problems for linearized Schródinger equation. The coefficients $\Psi_{n,k,\chi}$, $\chi \in K'_{n,k}$ are determined from algebraic equations. The set $K'_{n,k} = K_{n,k}\backslash\{\pm\varphi\}$. The phase set K_n for the $n-$th order term of the asymptotics as $\lambda \to \infty$ is determined by formula

$$K_1 = \pm\varphi; \quad K_2 = \pm\varphi, \pm S,$$
$$K_n = \cup_{j_1+j_2+j_3=n}\chi_{j_1} + \chi_{j_2} + \chi_{j_3}, \ \ , \chi_{j_k} \in K_{j_k}.$$

At last the post-resonant expansion has the following:



0.4
0.2
U
0
-0.2
-0.4
-40
-20
0
x
20
40
100
150
200
t
250
300

Higher-order terms and matching

The structure of constructed asymptotic solution when $l < 0$ and $l > 0$ are sufficiently different. We concentrate on the description of the changing of the solution from the pre-resonant to post-resonant form. This transition takes place in the thin layer near the curve $l = 0$. In this transition layer the amplitude of the solution increases due to the resonant pumping. The value of the amplitude is defined by the width of the resonant layer. We found the width of the layer by construction and analysis of the higher-order terms of the asymptotic solution in all domains. This analysis looks very complicated but it is necessary to match the asymptotics of the solution in different domains and obtain formula (13). This formula defines the leading order term of the solution after the slowly passage through the resonance.

Main result

Let us formulate the main result of the work. If the solution of (1) has the form

$$U \sim -\varepsilon^2 \frac{f}{l} \exp(iS(t_2, x_2)/\varepsilon^2) + c.c.,$$

when $l < 0$, then in the domain $l > 0$ this asymptotic solution is

$$U(x,t,\varepsilon) \sim \varepsilon \Psi(x_1, t_1, t_2) \exp\{i\varphi(x_2, t_2)/\varepsilon^2\} + c.c. \quad (12)$$

The phase function φ satisfies the eikonal equation

$$(\partial_{t_2}\varphi)^2 - (\partial_{x_2}\varphi)^2 - 1 = 0$$

with conditions

$$\varphi|_{l=0} = S|_{l=0}, \quad \partial_{t_2}\varphi|_{l=0} = \partial_{t_2} S|_{l=0}.$$

The envelope function of the leading-order term is a solution of the nonlinear Schródinger equation

$$2i\partial_{t_2}\varphi\partial_{t_2}\Psi + \partial_\xi^2\Psi + i[\partial_{t_2}^2\varphi - \partial_{x_2}^2\varphi]\Psi + \gamma|\Psi|^2\Psi = 0,$$

where the ξ is defined by

$$\frac{dx_1}{d\xi} = \partial_{t_2}\varphi, \quad \frac{dt_1}{d\xi} = \partial_{x_2}\varphi.$$

The initial condition for Ψ is

$$\Psi|_{l=0} = \int_{-\infty}^{\infty} d\sigma f(x_1) \exp(i\int_0^\sigma d\mu \lambda(x_1, t_1, \varepsilon)), \quad (13)$$

The integration in this integral is done in the characteristic direction related with the equation for $W_{1,1}$.

Nonlinear Schrödinger equation (NLSE) is a mathematical model for wide class of wave phenomenons from the signal propagation in optical fiber to the surface wave propagation. This equation can be considered as an ideal model equation. Here we consider the NLSE perturbed by the small driving force.

$$i\partial_t\Psi + \partial_x^2\Psi + |\Psi|^2\Psi = \varepsilon^2 f e^{iS/\varepsilon^2}, \qquad 0 < \varepsilon \ll 1. \quad (14)$$

The following picture shows the scattering process of one soliton to two solitons for equation (14), where amplitude of external force $\varepsilon = 0.1$, $S = 0.005t^2$,

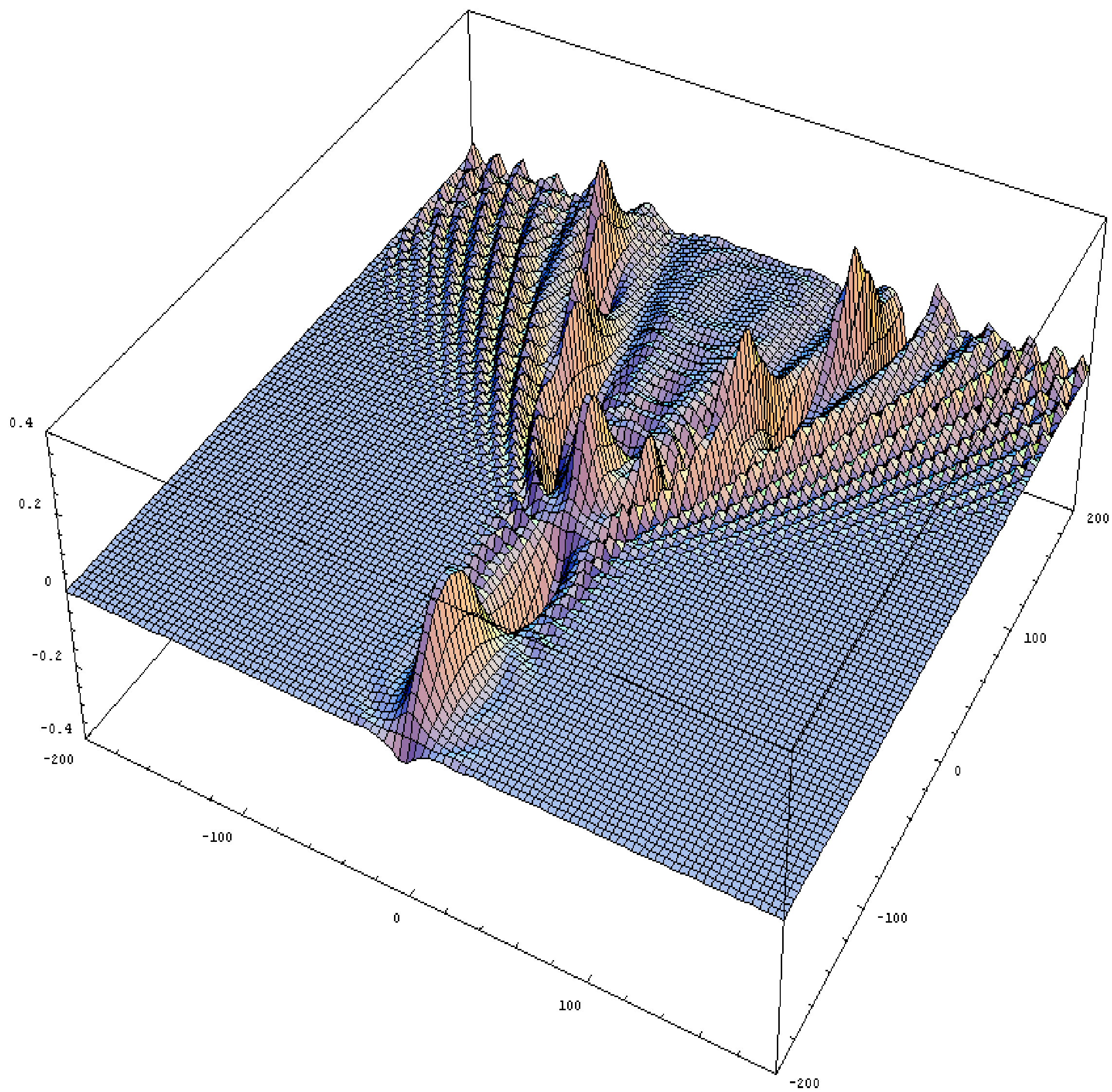
$$f = 2\sqrt{2}\cosh^{-1}(0.2x) + 2\sqrt{2}\exp(0.2ix)\cosh^{-1}(0.2x-5) +$$

$$2\sqrt{2}\exp(-0.2ix)\cosh^{-1}(0.2x+5),$$

initial data is pure soliton of NLSE at

$$\Psi|_{t=-200} = 0.2\sqrt{2}\cosh^{-1}(0.2x),$$

resonance curve is $t = 0$.



0.4
0.2
0
-0.2
-0.4
-200
-100
0
100
200
-200
-100
0
100
200

For numeric justification of our results we obtain the annihilate of the soliton on a local resonance. In this section we justify our asymptotic formula for the NLSE. Let us consider the pure soliton initial condition for equation (14):

$$\Psi(x,t,\varepsilon)|_{t=t_0}=\frac{2\sqrt{2}\varepsilon\eta\,\text{exp}\{-i2c\varepsilon x-4(c^2-\eta^2)t_0\varepsilon^2\}}{\cosh(2\eta\varepsilon x+8c\eta\varepsilon^2t_0+s)}$$

According of our analytical results this initial condition leads to one soliton solution as the leading-order term of the asymptotic solution:

$$\overset{1}{u}(x_1,t_2)=\frac{2\sqrt{2}\eta\,\text{exp}\{-i2cx_1-4i(c^2-\eta^2)t_2\}}{\cosh(2\eta x_1+8c\eta t_2+s)}.$$

This soliton propagates up to the resonance curve $t=0$.

To annihilate this soliton on the resonance curve one may choose the specific form of the amplitude of the perturbation. This form of the perturbation is defined by the formula:

$$0=\overset{1}{u}(x_1,0)+(1-i)\sqrt{\pi}f(x_1).$$

Hence

$$f(x_1)=\frac{-(1+i)}{2\sqrt{\pi}}\overset{1}{u}(x_1,0)=\frac{-(1+i)}{\sqrt{\pi}}\frac{\sqrt{2}\eta\,\text{exp}\{-i2cx_1\}}{\cosh(2\eta x_1+s)}$$

To illustrate this by numerical simulations we choose $\varepsilon=0.1,\eta=1,s=0,c=0,t^0=-200$. Then the original equation (14) has the form

$$i\partial_t\Psi+\partial_x^2\Psi+|\Psi|^2\Psi=0.01f\,\text{exp}\{i0.005t^2\}.$$

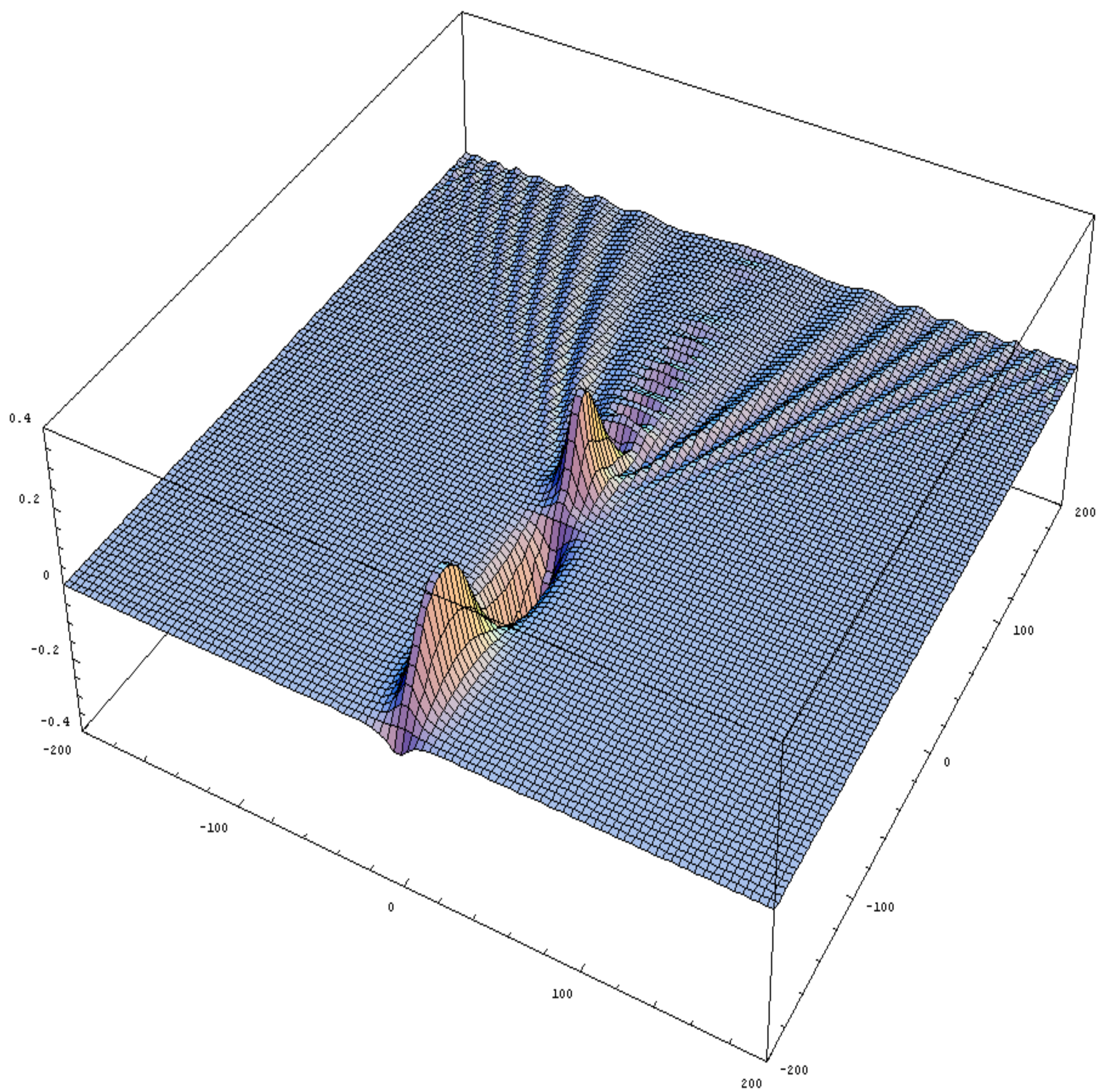
Initial condition is

$$\Psi|_{t=-200} = \frac{0.2\sqrt{2}}{\cosh(0.2x)},$$

and amplitude of the perturbation is

$$f = \frac{-(1+i)}{\sqrt{\pi}} \frac{\sqrt{2}}{\cosh(0.2x)}$$

The numerical simulations of annihilation process for soliton of NLSE are presented on the following figure. This justifies the formulas obtained above by matching method.



References

S.G.Glebov, O.M. Kiselev, V.A. Lazarev. Slow passage through resonance for a weakly nonlinear dispersive wave. SIAM Journal of Applied Mathematics, accepted.

S.G.Glebov, O.M. Kiselev,The slowly passage through the resonances and wave packets with the different carriers. ArXiv: math-ph 0503019. (submitted to Dynamics of Partial Differential Equations)

S.G. Glebov, O.M. Kiselev. The stimulated scattering of solitons on a resonance. Journal of Nonlinear Mathematical Physics, 2005, v.12, n3, p.330-341.

S.G.Glebov, V.A.Lazarev, O.M. Kiselev. Generation of solitary packets of waves by resonance. Proceedings of International seminar "Day on Diffraction-2003", SPb. Russia. pp.46-51.

S.G.Glebov, O.M. Kiselev, V.A.Lazarev. Birth of solitons during passage through local resonance. Proceedings of the Steklov Institute of Mathematics. Suppl., 2003, issue 1, S84-S90.