Hard loss of stability in Painlevé-2 equation

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Preface

The purpose of my talk is to show constructing of an uniform asymptotic solution of the Painlevé-2 equation:

$$V'' + 2V^3 + zV = \alpha, \quad \text{when} \quad z \to \infty, \;\; \alpha \to \infty. \tag{1}$$

Beginning with works by P.Boutroux (1913, 1914) to the present day constructing of asymptotic solutions for the Painlevé equations is a popular direction in a nonlinear analysis. To prove this sentence I can mention the works by R. Haberman, A.R. Its, N. Joshi, A.A. Kapaev, A.V. Kitaev, M. Kruskal, V.Yu. Novokshenov but even this list is far from completeness.

We will be interested the in works concerning an elliptic asymptotics for Painlevé-2 equation. The first asymptotics in this field was constructed by P.Bouroux. More later, these asymptotics have been justified with a Monodromy-Preserve Method (see review devoted to these problem by A.V. Kitaev in Usp. Mat. Nauk, 1994). But the asymptotics having been constructed are nonuniform with respect to two parameters of the Painlevé-2 equation (z, α). At beginning of 20th century M.Painlevé found that the nonlinear equations called Painlevé equations now may be considered as some scaling limits from each other. Now this property of the equations has been studied for solutions too. Main stream of this study direct to the Monodromy-Preserve Method. This gives to us an formal description of the solutions in the terms of monodromy data and allows to study the asymptotic behaviour of the solution in some intervals of parameters (z, α), but not uniformly.

Naive statement of the problem

It will be more convenient to rewrite this equation in the form obtained after substitutions $V = \alpha^{2/3}u$, $z = \alpha^{1/3}t$, $\varepsilon = 1/\alpha$:

$$\boldsymbol{\varepsilon^2 u'' + 2u^3 + tu = 1}, \quad \text{where} \quad \boldsymbol{0 < \varepsilon \ll 1}. \tag{2}$$

Since the parameter ε is small we neglect the term $\varepsilon^2 u''$ from the equation. Let us consider an obtained cubic equation:

$$\boldsymbol{2u^3 + tu = 1}. \tag{3}$$

The discriminant of the equation (3) has the form:

$$D = \Big(\frac{t}{6}\Big)^3 + \Big(\frac{1}{4}\Big)^2.$$

The discriminant $D < 0$ when $t < t_*$ and hence the cubic equation has three real roots $u_1(t) < u_2(t) < u_3(t)$. If $t > t_*$ then $D > 0$ and the cubic equation has one real and two complex conjugate roots. At $t = t_* = -3\,2^{-1/3}$ the roots $u_1(t)$ and $u_2(t)$ merge $u_1(t_*) = u_2(t_*) = u_* = -4^{-1/3}$.

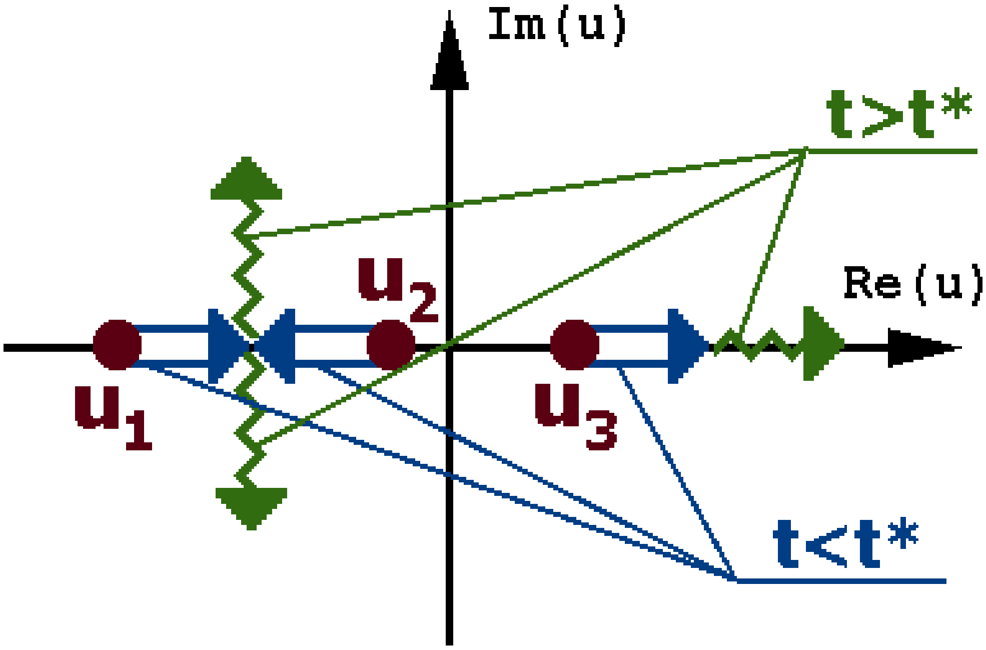


Figure 1: **The roots of the cubic equation.**

When $t < t_*$ it is possible to construct a real formal solution of (2):

$$u(t, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^{2k} \overset{2k}{u}(t) \qquad (4)$$

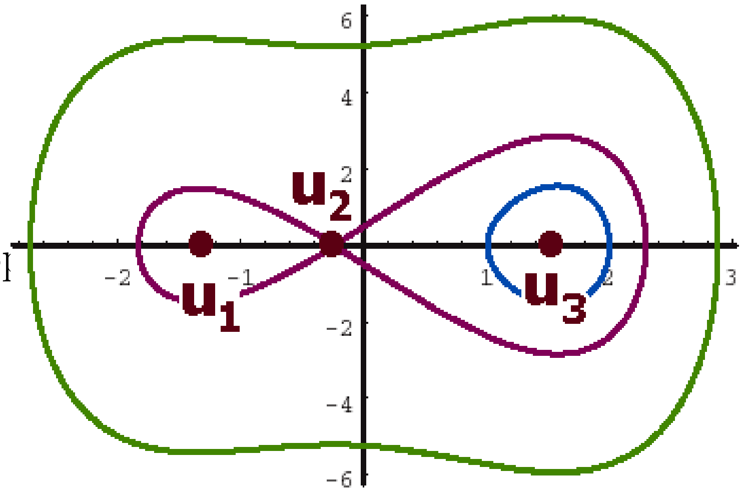
by taking as a leading term $\overset{0}{u}(t)$ any of roots $u_j(t)$.

Our goal is to construct a smooth asymptotic solution of the equation (2) on a segment $[t_* - a, t_* + a]$, $\quad a = const > 0$ with the leading term $\overset{0}{u} = u_1(t)$ when $t < t_*$.

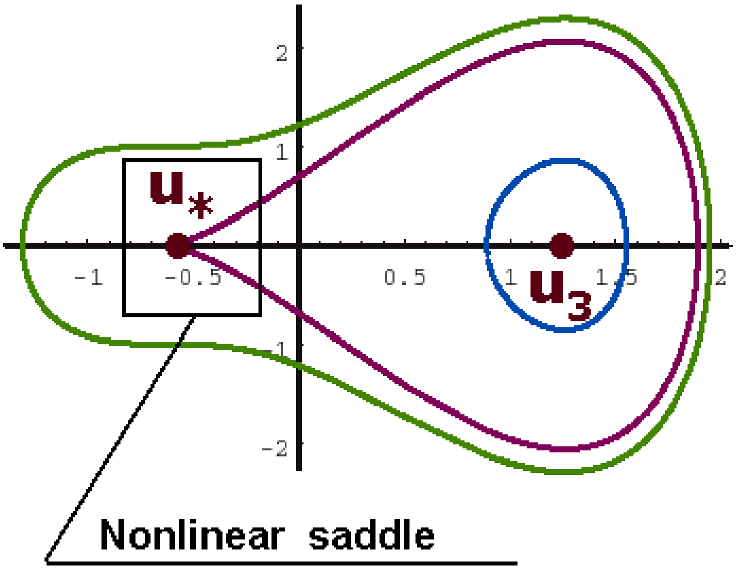
Qualitative analysis To understand a behaviour of the asymptotic solution after the bifurcation point $t > t_*$ we will study the qualitative behaviour of the solutions for the differential equation with freezing coefficient. Let us consider the equation:

$$\varepsilon^2 \mathbf{u}'' + 2\mathbf{u}^3 + \mathbf{T}\mathbf{u} = \mathbf{1}. \tag{5}$$

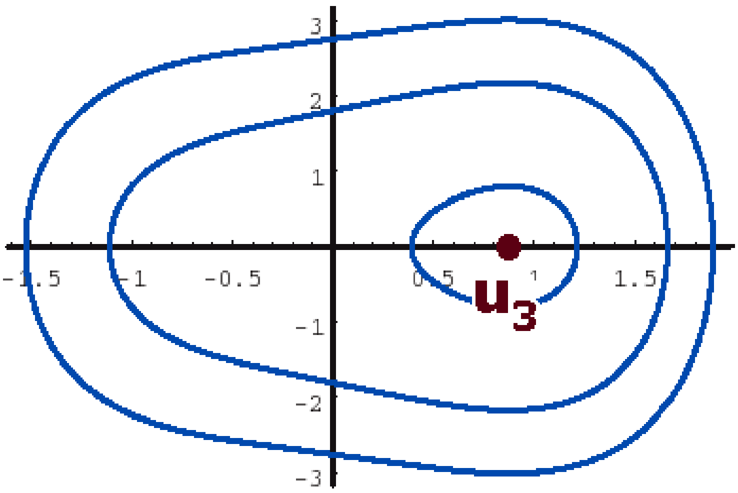
This equation has different phase portraits when $T < t_*$, when $T = t_*$ and when $T > t_*$.



If $T < t_*$ then the phase portrait of the equation (5) has three equilibrium u_1 and u_3 are stable and u_2 is instable equilibrium. We can see two separatrix loo in this picture.



t $T = t_*$ the phase portrait of (5) has wo equilibriums. u_* is instable and ι_3 is stable. In this picture we see only ne separatrix. u_* is a nonlinear addle point.

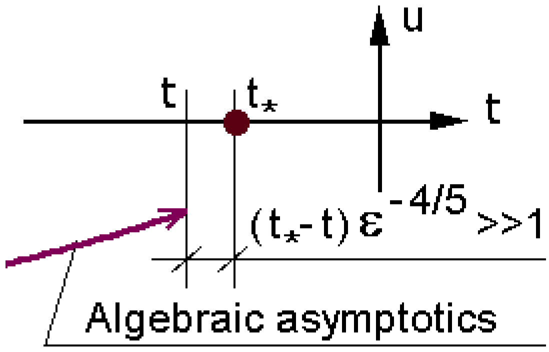


At last you can see the phase portrait of when $T > t_*$ In this case we see only one equilbrium u_3 and periodic trajectories

Conclusion. **The qualitative analysis for the equation with frozen coefficient shows that asymptotic solution we construct will vary slowly when $t < t_*$ and will fast oscillates when $t > t_*$.**

Asymptotic analysis

Here we show the main result an asymptotic analysis for uniform asymptotic of the Painlevé-2 equation.



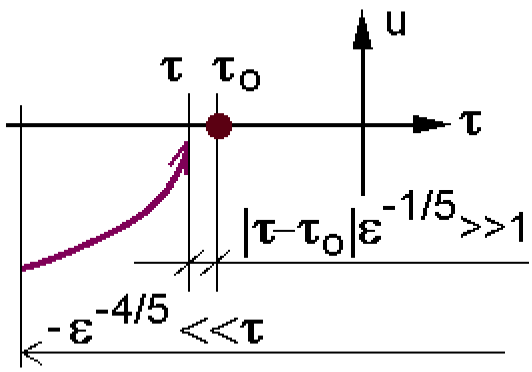
When $t < t_* - a$, $(a = \text{const} > 0)$ and $(t_* - t)\varepsilon^{-4/5} \gg 1$ the asymptotic solution with respect to $\operatorname{mod}(O(\varepsilon^6) + O(\varepsilon^6(t - t_*)^{-13/2}))$ has the form

$$u(t,\varepsilon) = u_1(t) + \varepsilon^2 \frac{-2tu_1(t)}{(6u_1^2(t) + t)^4} + \varepsilon^4 \overset{2}{u}(t).$$

The last term of the formal asymptotic solution as $t \to t_* - 0$ can be written as:

$$\overset{2}{u}(t) = O((t - t_*)^{-9/2}).$$

When $|t - t_*| \ll 1$ the asymptotic solution is defined by two different types of the formal asymptotic expansions. The first one has the form



$$u(t,\varepsilon) = u_* + \varepsilon^{2/5} \overset{0}{v}(\tau) + \varepsilon^{4/5} \overset{1}{v}(\tau). \tag{6}$$

Here the variable τ is defined by the formula $\tau = (t - t_*)\varepsilon^{-4/5}$, the function $\overset{0}{v}(\tau)$ is defined as the solution of the equation Painlevé-1 (see R. Haberman, 1979):

$$\frac{d^2 \overset{0}{v}(\tau)}{d\tau^2} + 6u_* \overset{0}{v}{}^2 + u_*\tau = 0, \quad \overset{0}{v}(t) = -\sqrt{\frac{-\tau}{6}} + O(\tau^{-2}),\ \tau \to -\infty.$$

The formula (6) is asymptotic solution with respect to $\operatorname{mod}(O(\varepsilon^{8/5}\tau^2) + O(\varepsilon^{8/5}))$ as $1 \ll -\tau \ll \varepsilon^{-4/5}$.

The function $\overset{0}{v}(\tau)$ has poles of second order at some points $\tau_k,\ k = 0, 1, 2, \ldots$. Near the poles the last term of the asymptotic solution can be written as

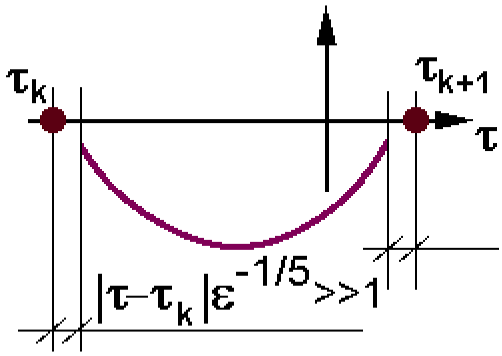
$$\overset{1}{v}(\tau) = O((\tau - \tau_k)^{-4}), \quad \text{as } \tau \to \tau_k.$$

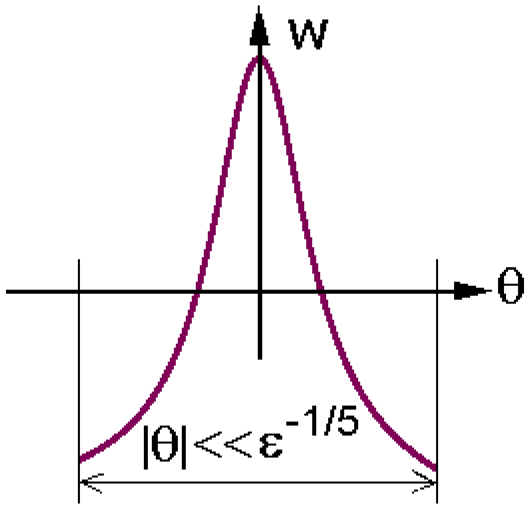
The expansion (6) is suitable at $\varepsilon^{-1/5}|\tau - \tau_k| \gg 1$. The formula (6) is asymptotic solution with respect to $\operatorname{mod}(O(\varepsilon^{8/5}) + O(\varepsilon^{8/5}\tau^2) + O(\varepsilon^{8/5}\tau_k(\tau - \tau_k)^{-8}))$.

As $\tau \to \infty$ the last term of the asymptotics is

$$\overset{1}{v}(\tau) = O(\tau^{9/4}) + O\Big(\frac{\tau^{9/4}}{(\tau - \tau_k)^4}\Big) \quad \text{as } \tau \to \infty \text{ and } \tau \neq \tau_k.$$

The formula (6) is asymptotic solution with respect to mod$(O(\varepsilon^{8/5}) + O(\varepsilon^{8/5}\tau^{9/2}) + O(\varepsilon^{8/5}\tau_k^{9/2}(\tau-\tau_k)^{-8}))$. The expansion (6) is suitable as $\tau \ll \varepsilon^{-8/35}$ and $\varepsilon^{-1/5}|\tau-\tau_k|\tau_k^{-7/8} \gg 1$.





The second one, which is valid in the neighborhoods $|\tau-\tau_k||\tau_k|^{1/5} \ll 1$ of poles τ_k of the function $\overset{0}{v}(\tau)$, reads as

$$u(t,\varepsilon) = u_* + \overset{0}{w}(\theta) + \varepsilon^{4/5}\overset{1}{w}(\theta), \qquad (7)$$

where $\theta = (\tau-\tau_k)\varepsilon^{-1/5}$. The main term is (see also R. Haberman, 1979):

$$\frac{a^{-}\, w}{d\theta^2} + 6u_* \overset{0}{w}{}^2 + 2\overset{0}{w}{}^3 = 0, \ \textbf{where} \ \overset{0}{w}(\theta) = \frac{-16u_*}{4+16u_*^2\theta^2}.$$

The last term of the formal asymptotics (7) at $|\theta| \to \infty$ can be written as $\overset{1}{w}(\theta) = O(\theta^2|\tau_k|)$. The formula (7) is asymptotic solution with respect to mod$(O(\varepsilon^{8/5}) + O(\varepsilon^{8/5}\theta^4\tau_k^2))$.

The constructed special solution of the Painlevé-1 equation may be represented as

$$\overset{\mathbf{0}}{\mathbf{v}}(\tau) \sim \sqrt{\tau}\,\mathcal{P}\Big(\frac{\mathbf{4}}{\mathbf{5}}\tau^{\mathbf{5/4}}, \mathbf{1}, \mathbf{g3}\Big)\ \textbf{as}\ \tau \to \infty,$$

where $\mathcal{P}(S)$ is solution of Weierstrass equation. Hence the poles of $\overset{0}{v}(\tau)$ concentrate as $\tau \to \infty$. Therefore the combined asymptotics is satisfactory as long as $\tau_k \ll \varepsilon^{-4/5}$.

When $(t-t_*)\varepsilon^{-2/3} \gg 1$ and $t < t_* + a$ the asymptotic solution with respect to mod$(O(\varepsilon^2) + O(\varepsilon^2(t-t_*)^{-3/2}))$ has the fast oscillating behavior(Kuzmak, 1959):

$$u(t,\varepsilon) = \overset{0}{U}(t_1,t) + \varepsilon\,\overset{1}{U}(t_1,t).$$

Here the last term of the asymptotics (8) at $t \to t_* + 0$ can be written as:

$$\overset{1}{U}(t_1,t)=O((t-t_*)^{-3/2}).$$

The leading term of the asymptotic solution satisfies the Cauchy problem:

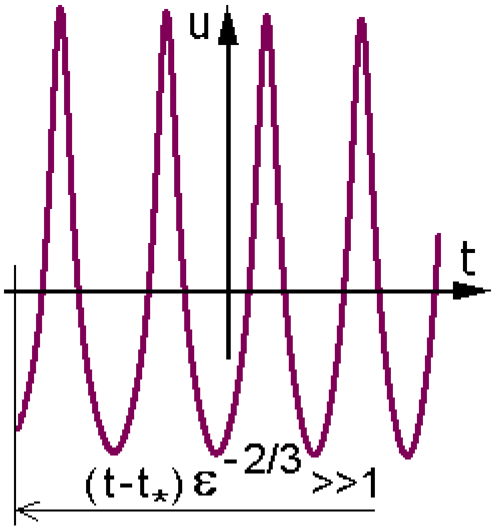
$$(S')^2(\overset{0}{U}_{t_1})^2=-\overset{0}{U}{}^4-t\,\overset{0}{U}{}^2+2\,\overset{0}{U}+E(t),$$

$$\overset{0}{U}|_{t_1=0}=u_*.$$

Here $t_1=S(t)/\varepsilon$. The function $E(t)$ is defined by the equation

$$\int_{\beta(t)}^{\alpha(t)}\sqrt{-x^4-tx^2+2x+E(t)}dx=\pi,$$

where $\alpha(t)$ and $\beta(t)$ $(\alpha(t)>\beta(t))$ are two real roots of the equation $-x^4-tx^2+2x+E(t)=0$, other roots of this equation are complex.



The phase function $S(t)$ is the solution of an equation :

$$T=S'\sqrt{2}\int_{\beta(t)}^{\alpha(t)}\frac{dx}{\sqrt{-x^4-tx^2+2x+E(t)}}.$$

Where T is the constant defined by the formula

$$T=\frac{\sqrt{2}C_*(k)}{2|u_*|^{1/2}}\Big(\frac{3}{6-2k^2}\Big)^{1/4},$$

$k\approx 0.463$ is the unique solution of the equation

$$\int_0^\infty dy\frac{-ky+k^2+1}{[(y-k)^2+1]^{5/2}}y^{5/2}=0,\ \text{and } C_*(k)=\int_0^\infty\frac{dy}{\sqrt{y[(y-k)^2+1]}}.$$

Remark 1. The domains of validity of the algebraic asymptotic solution and the asymptotic solution (6) intersect, so that these expansions match. The solution of the Painlevé-1 equation defining the asymptotics (6) has infinite sequence of the poles τ_k, $k=1,2,\dots$. Near all of these poles we match the asymptotic solutions (6) and (7). As the number of the pole $k\to\infty$ the domain of validity of this combined asymptotics (6) and (7) intersects with the domain of validity for the fast oscillating asymptotic solution (8). It allows to match the combined asymptotics with the fast oscillating asymptotic solution.

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Saddle-center bifurcation in the Painleve-2 equation

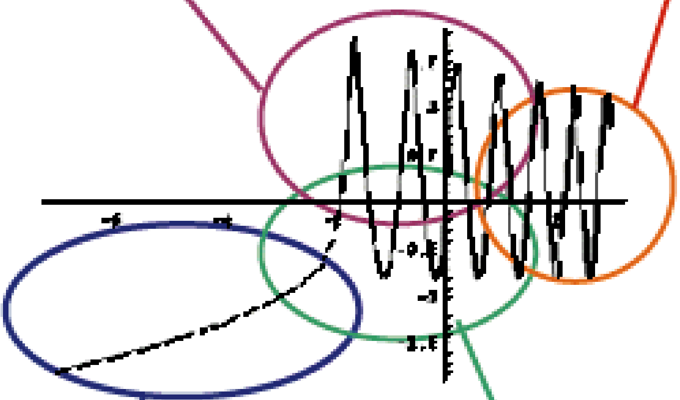
Separatrix layers : $\theta = (\tau - \tau_\kappa)\epsilon^{-1/5}$,

$$u(t,\epsilon) = u_* + w(\theta) + O(\epsilon^{4/5}\theta^2|\tau_k|).$$

Fast Oscillating asymptotics:

$$u(t,\epsilon) = U\left(S(t)\big/ \epsilon , t\right) + O(\epsilon),$$

at $(t - t_*)\epsilon^{-2/3} >> 1.$



Painleve layers: $\tau = (t - t_*)\epsilon^{-4/5}$,

$$u(t,\epsilon) = u_* + \epsilon^{2/5} v(\tau) + \mathrm{O}(\epsilon^{4/5}\tau) +$$

$$+ \mathrm{O}((\tau - \tau_k)^{-4}\epsilon^{4/5}).$$

Algebraic asymptotics:

$$u(t,\epsilon) = u_0(t) + \epsilon^2 u_1(t) + O(\epsilon^4 (t - t_*)^{-9/2}),$$

at $(t_* - t)\epsilon^{-4/5} >> 1.$