Stochastic properties of two-wheeled robot on a soft surface

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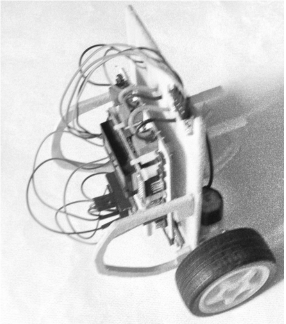

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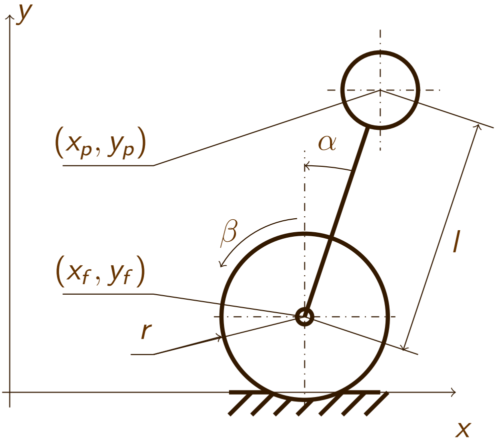
Outline

Wheeled inverted pendulum (WIP)



The mass of the pendulum is defined as m.

The mass of the rim is defined as M.

$\rho = M/m$ is small.

$\zeta = l/r$ is large.

Problems

- The stability of the of the wheeled pendulum with "bad" gearboxes, when one should apply an initial torque to rotate the wheel.
- The stability on of the wheeled pendulum on a soft surface.
- The stochastic properties of the data form the sensors.
- The control using the stochastic data.
- Dynamics under the stochastic controller.

Special cases

- The simplest case is the inverted pendulum ($\rho = 0$):

$$\ddot{\alpha} = \sin(\alpha).$$

  This case is unstable at the upper position ($\alpha = 0$).

- The equation for the WIP on the hard horizontal line is following:

$$(\sin^2(\alpha)\zeta + 2)\zeta\rho\ddot{\alpha} = \zeta\rho(\zeta + 2)\sin(\alpha) - \frac{1}{2}\dot{\alpha}^2\zeta^2\rho\sin(2\alpha) \\ -2(\zeta\cos(\alpha) + \rho^2(\zeta + 2))\,u.$$

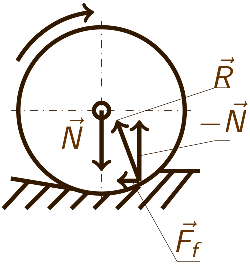
The wheel control torque.

The wheel control torque will used to stabilize the pendulum. See:

A.M. Formalskii.*Stabilisation and Motion Control of Unstable Objects.* Series:De Gruyter Studies in Mathematical Physics 33, 2016

C. R. Halkyard R. P. M. Chan, K. A. Stol. *Review of modelling and control of two-wheeled robots.* Annual Reviews in Control, 37:89–103, 2013.

A resistance of rotation on a soft surface.



$\vec{N}$ –the weight;
$\vec{F}_f$ –the friction;
$\vec{R}$ –the reaction;

Figure: Details see for ex. A. Yu. Ishlinskii and D. D. Ivlev, Mathematical Theory of Plasticity [in Russian], Fizmatlit, Nauka, 2001.

If $\dot{\beta} \neq 0$ then the resistance of rotation: $M_f = -\nu \operatorname{sgn}(\dot{\beta})$;
If $\dot{\beta} = 0$ then the resistance of rotation can be included into the region: $M_f \in (-\nu, \nu)$.

A differential inclusion for dynamics

$$
\begin{aligned}
\ddot\alpha &= \sin(\alpha) - (\cos(\alpha - z)\ddot\beta + \sin(\alpha - z)\dot\beta^2)\rho - 2\frac{\rho}{\zeta}u, \\
(\zeta + 2)\rho\ddot\beta &\in F(\alpha, \dot\alpha, \ddot\alpha, \dot\beta).
\end{aligned} \tag{1}
$$

The map $F(\alpha, \dot\alpha, \ddot\alpha, \dot\beta)$ has the form:

$$
F(\alpha, \dot\alpha, \ddot\alpha, \dot\beta) = \begin{cases} f - \nu\ \mathrm{sgn}(\dot\beta), & \{\forall\dot\beta \neq 0\}; \\ (-\nu, \nu), & \{\dot\beta = 0\} \cup \{|f| \le \nu\}; \\ f, & \{\dot\beta = 0\} \cup \{\{\alpha, \dot\alpha, \ddot\alpha\} \in \{|f| > \nu\}\}. \end{cases}
$$

$$
f = -\sin(z) - \left(\ddot\alpha\cos(\alpha - z) - \dot\alpha^2\sin(\alpha - z)\right)\zeta + \frac{2}{\rho}u.
$$

- z is the current inclination of the soft surface;
- ν is the torque of the friction resistant.

Two stable branches of the solutions

The control torque with the PID controller is:
$u = k_1\alpha + k_2\dot\alpha + k_3 A$, where, $A \equiv \int^t \alpha(t)dt$. In this case system (1) has a particular solution:

$$\alpha \equiv 0, \qquad A = \frac{\operatorname{sgn}(\dot\beta)\zeta\nu\rho}{(2k_3\zeta + 4k_3)\,\rho^2 + 2k_3\zeta},$$

$$\beta = \begin{cases} \beta_0 + \beta_1(t-t_0) - \dfrac{\zeta\nu\rho\operatorname{sgn}(\dot\beta)}{(2k_3\zeta + 4k_3)\,\rho^2 + 2k_3\zeta}\dfrac{(t-t_0)^2}{2}, & (t-t_0) < T; \\ \beta_0 + \beta_1 T - \dfrac{\zeta\nu\rho\operatorname{sgn}(\dot\beta)}{(2k_3\zeta + 4k_3)\,\rho^2 + 2k_3\zeta}\dfrac{T^2}{2}, & (t-t_0) \ge T, \end{cases} \tag{2}$$

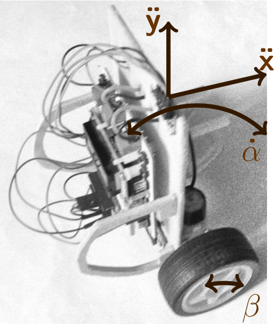
where

$$T = \frac{1}{\beta_1}\frac{\zeta\nu\rho\operatorname{sgn}(\dot\beta)}{(2k_3\zeta + 4k_3)\,\rho^2 + 2k_3\zeta}, \quad \{t_0, \beta_0, \beta_1\} \in \mathbb{R}.$$

Theorem

There exists the set of the parameters $\zeta, \rho, k_1, k_2, k_3$ when solution (2) is an attractor as $(t-t_0) < T$.

Observability and sensors



To use a feedback controller for stabilizing of the WIP one should obtain data of the sensors at the moment $t_i = t_{i-1} + dt$.
The gyroscope gives us the angle velocity of the pendulum $\dot{\alpha} \to a_1$;
The tilt sensor gives us the linear accelerations $(\ddot{x}, \ddot{y}) \to a_2$;
The encoder gives us the digital value of the wheel angle $\beta \to b_1 = (\beta_i - \beta_{i-1})/(dt)$.

$$\frac{a_2}{\cos(\alpha)} = -(\cos(\alpha - z)(\frac{\sin(\alpha)\gamma}{\cos(\alpha)\rho} - \frac{a_2\sin(\alpha)}{\cos(\alpha)\rho} + b_2) + b_1^2\sin(\alpha - z))\rho - \frac{2u\rho}{\zeta} + \sin(\alpha), \quad (3)$$

$$(\zeta + 2)(\frac{\sin(\alpha)\gamma}{\cos(\alpha)\rho} - \frac{a_2\sin(\alpha)}{\cos(\alpha)\rho} + b_2)\rho \in \left\{\begin{array}{cc} f - \nu\mathrm{sgn}(b_1), & b_1 \neq 0; \\ (-\nu, \nu), & b_1 = 0 \cup |f| < \nu; \\ f, & b_1 = 0 \cup |f| \geq \nu; \end{array}\right. \quad (4)$$

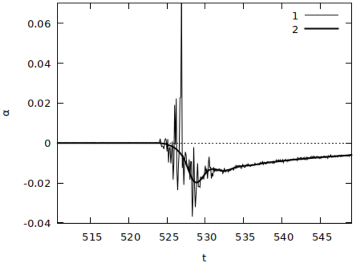
$$f = -\sin(z) - (a_2\cos(\alpha - z) - a_1^2\sin(\alpha - z))\zeta + \frac{2}{\rho}u.$$

Theorem about the observability

Theorem

Let one know the values of the acceleration $(\ddot{x}, \ddot{y})$, angle velocity of the pendulum $\dot{\alpha}$ and angle velocity of the wheel $\dot{\beta}$, then the observed dynamical system is solution of the trigonometric equation (3) and the inclusion (5).

Observed data



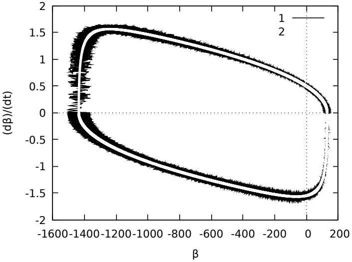
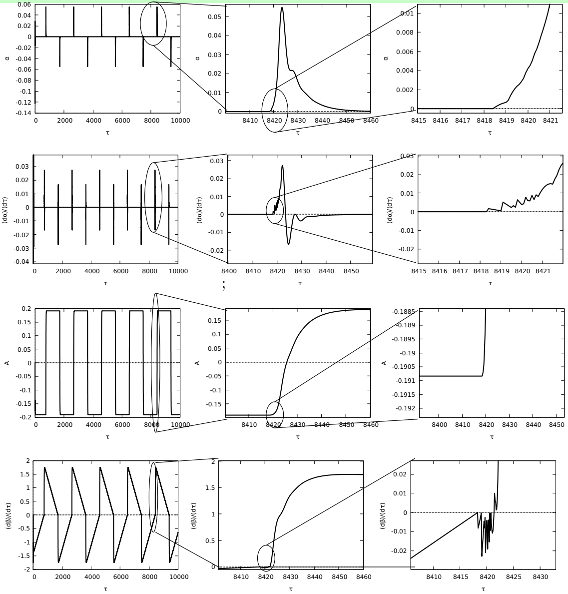


Figure: The observed data on the plane (t, α) and on the phase plane $(\beta, \dot{\beta})$ as $z \equiv 0$ for the WIP on the soft surface. The feedback controller use the tilt sensor and the gyroscope. The parameters of the system are following: $\rho = 0.2$, $\zeta = 10$, $\nu = 0.05$,$\gamma = 1$, the PID coefficients are: $k_1 = 1.7$, $k_2 = 0.2$, $k_3 = 0.02$. The relative errors are uniform distributed data at the interval $(-0.02, 0.02)$. The dynamic system (1) solved at $A \sim 0.2385$, $\alpha = 0.02$, $\dot{\alpha} = 0$, $\beta = 0, \dot{\beta} = 0.5$ by Runge-Kutta method of fourth-order method with the step 0.1.

Dynamics on soft surface

Define δ the error of the observed data of $\dot{\beta}$.
The layer $|\dot{\beta}| \leq \max\{\delta\}$ appears in the stochastic system near the hyperplane $\dot{\beta} = 0$. In this layer the term $\nu \mathrm{sgn}(\dot{\beta})$ takes the random values $\pm\nu$ at $t \in (t_i, t_{i+1})$.
There exists the small neighbourhood $(\Delta_\pm)$ near the unstable lines $(A_\pm, 0, 0, 0, \dot{\beta})$, where can be obtained four typical cases:

Trajectories of the WIP



- Let $\dot\beta > 0$, $A < A_+$
  - and $\mathrm{sgn}\breve{\dot\beta} = 1$, then the trajectory is kept in the neighbourhood of the line $(A_+, 0, 0, 0, \dot\beta)$;
  - and $\mathrm{sgn}\tilde{\dot\beta} = -1$, then the trajectory is kept in the neighbourhood of the line $(A_+, 0, 0, 0, \dot\beta)$.
- Let $\dot\beta < 0$, $A > A_-$
  - and $\mathrm{sgn}\breve{\dot\beta} = -1$, then the trajectory is kept in the neighbourhood of the line $(A_-, 0, 0, 0, \dot\beta)$;
  - and $\mathrm{sgn}\breve{\dot\beta} = 1$, then the trajectory is kept in the neighbourhood of the line $(A_-, 0, 0, 0, \dot\beta)$.

The sequence of the changes of the trajectories at the neighbourhoods of the lines $(A_\pm, 0, 0, 0)$ leads to the appearance of the hysteresis loop at the phase plane $(\beta, \dot\beta)$, see figure 2.

Phase portrait under PID-controller

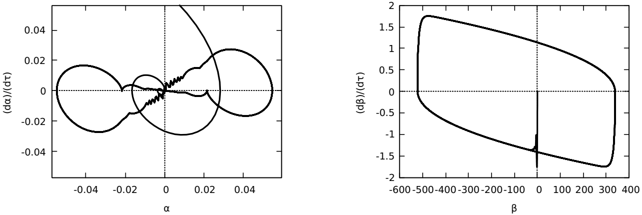


Figure: Here one can see the solution of (1) with the initial conditions $\tau = 0$, $\alpha = -0.1, \dot{\alpha} = -0.1, \beta = 0, \dot{\beta} = -0.1$ and under the proportional-integral-derivation controller, where $k_1 = 1.5, k_2 = 0.2, k_3 = 0.05$. On the left-hand side it is shown the limit cycle on the phase plane $(\alpha, \dot{\alpha})$ and on the right-hand side it is shown the limit cycle on the phase plane $(\beta, \dot{\beta})$. The parameters of the pendulum are following: $\zeta = 10$, $\rho = 0.2$ and $\nu = 0.1$.

The stochastic data for the PID-controller

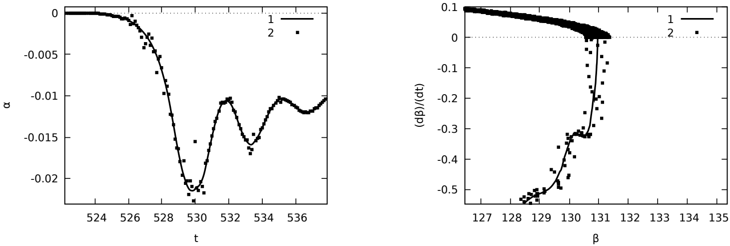


Figure: In this picture one can see the result of the numeric modelling of stochastically controlled WIP at $\xi = 10$, $\rho = 0.2$, $\nu = 0.05$, $\gamma = 1$, $k_1 = 1.7$, $k_2 = 0.2$, $k_3 = 0.02$. On the left picture the plane (t, α) and the right one shows the plane $(\beta, \dot\beta)$. The step of the change of the control torque is 0.1. The line is the solution under the discrete control. The results of the measurements $\breve{\ddot\alpha}_i$, $\breve{\dot\alpha}_i$ and $\breve{\dot\beta}_i$ are modelled by the current values $\ddot\alpha_i$, $\dot\alpha_i$ and $\dot\beta_i$ with the uniform distribution of the relative errors at the interval $(-0.003, 0.003)$. The value of the angle $\breve\alpha_i$ is defined as the observed data. The value $\breve{A}$ is computed using $\breve\alpha$, $\breve{\dot\alpha}$ integrating by the trapezoidal method. It allows to obtain the control torque u_{i+1} at the interval $t \in (t_i, t_{i+1})$, where $t_{i+1} - t_i = \Delta t$. At $t \in (t_i, t_{i+1})$ the system for the WIP is solved for the constant value the control torque $u = u_{i+1}$ by the Runge-Kutta method of 4-th order and step 0.01.

Average time at $\alpha \equiv 0$

The time between the sequence measurements is equal Δt. Let the trajectory be in the neighbourhood $\Delta_\pm$ of the unstable line. The probability of $\mathrm{sgn}(\delta) = \pm 1$ in primary order as $\Delta_\pm \to 0$ equals $p_\pm \sim 1/2$. The average time for trajectory in this neighbourhood is following:

$$T_0 = \Delta t \sum_{n=1}^{\infty} \frac{n}{2^n} = 2\Delta t.$$

Theorem

The average time spending at Δ-neighbourhood of the unstable lines $(A_\pm, 0, 0, 0, \dot\beta)$ for the stochastic system (1) equals $2\Delta t$, where Δt is the time between the sequenced measurements of the state for the system.

Experimental data

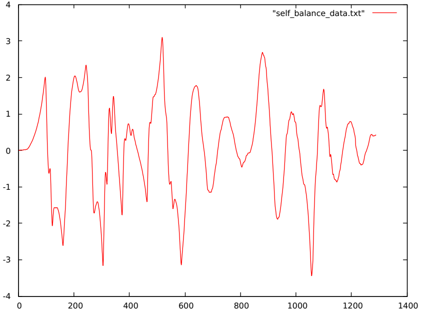
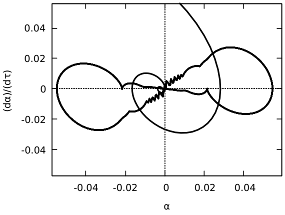
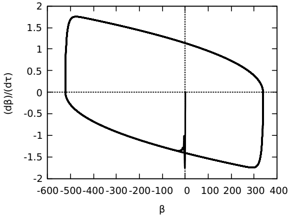


Figure: The experimental data from the wheeled robot with "bad" gear-box under PID-controller.

Conclusions

- If the robot contains the gyroscope, the tilt sensor and the encoder, then the system is observable.
- On the soft surface PID-controller stabilizes of the WIP on the stable limit circle near the upper position of the pendulum.





The bibliography

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