Stochastic properties of a segway on a soft surface

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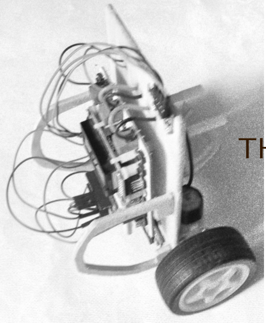

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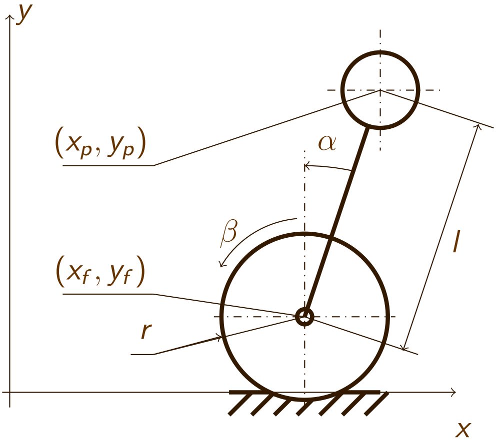
THE INTERNATIONAL CONFERENCE
ON DIFFERENTIAL EQUATIONS
AND
DYNAMICAL SYSTEMS

Outline

Wheeled inverted pendulum (WIP)



The mass of the pendulum is defined as m.

The mass of the rim is defined as M.

$\rho = M/m$ is small.

$\zeta = l/r$ is large.

Problems

- The stability of the of the wheeled pendulum with "bad" gearboxes, when one should apply an initial torque to rotate the wheel.
- The stability on of the wheeled pendulum on a soft surface.
- The stochastic properties of the data form the sensors.
- The control using the stochastic data.
- Dynamics under the stochastic controller.

Special cases

- The simplest case is the inverted pendulum ($\rho = 0$):

$$\ddot{\alpha} = \sin(\alpha).$$

This case is unstable at the upper position ($\alpha = 0$).

- The equation for the WIP on the hard horizontal line is following:

$$(\sin^2(\alpha)\zeta + 2)\zeta\rho\ddot{\alpha} = \zeta\rho(\zeta + 2)\sin(\alpha) - \frac{1}{2}\dot{\alpha}^2\zeta^2\rho\sin(2\alpha) \\ -2(\zeta\cos(\alpha) + \rho^2(\zeta + 2))\,u.$$

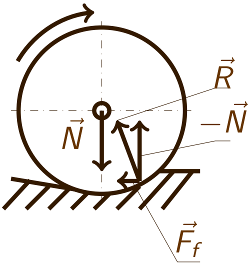
The wheel control torque.

The wheel control torque will used to stabilize the pendulum. See:

A.M. Formalskii.*Stabilisation and Motion Control of Unstable Objects.* Series:De Gruyter Studies in Mathematical Physics 33, 2016

C. R. Halkyard R. P. M. Chan, K. A. Stol. *Review of modelling and control of two-wheeled robots.* Annual Reviews in Control, 37:89–103, 2013.

A resistance of rotation on a soft surface.



$\vec{N}$ –the weight;
$\vec{F}_f$ –the friction;
$\vec{R}$ –the reaction;

Figure: Details see for ex. A. Yu. Ishlinskii and D. D. Ivlev, Mathematical Theory of Plasticity [in Russian], Fizmatlit, Nauka, 2001.

If $\dot{\beta} \neq 0$ then the resistance of rotation: $M_f = -\nu\,\mathrm{sgn}(\dot{\beta})$;
If $\dot{\beta} = 0$ then the resistance of rotation can be included into the region: $M_f \in (-\nu, \nu)$.

A differential inclusion for dynamics

$$\begin{aligned} \ddot{\alpha} &= \sin(\alpha) - (\cos(\alpha - z)\ddot{\beta} + \sin(\alpha - z)\dot{\beta}^2)\rho - 2\frac{\rho}{\zeta}u, \\ (\zeta + 2)\rho\ddot{\beta} &\in F(\alpha, \dot{\alpha}, \ddot{\alpha}, \dot{\beta}). \end{aligned} \quad (1)$$

The map $F(\alpha, \dot{\alpha}, \ddot{\alpha}, \dot{\beta})$ has the form:

$$F(\alpha, \dot{\alpha}, \ddot{\alpha}, \dot{\beta}) = \begin{cases} f - \nu \operatorname{sgn}(\dot{\beta}), & \{\forall \dot{\beta} \neq 0\}; \\ (-\nu, \nu), & \{\dot{\beta} = 0\} \cup \{|f| \leq \nu\}; \\ f, & \{\dot{\beta} = 0\} \cup \{\{\alpha, \dot{\alpha}, \ddot{\alpha}\} \in \{|f| > \nu\}\}. \end{cases}$$

$$f = -\sin(z) - \left(\ddot{\alpha}\cos(\alpha - z) - \dot{\alpha}^2\sin(\alpha - z)\right)\zeta + \frac{2}{\rho}u.$$

- z is the current inclination of the soft surface;
- ν is the torque of the friction resistant.

Two stable branches of the solutions

The control torque with the PID controller is:
$u = k_1\alpha + k_2\dot\alpha + k_3 A$, where, $A \equiv \int^t \alpha(t)dt$. In this case system (1) has a particular solution:

$$\alpha \equiv 0, \qquad A = \frac{\mathrm{sgn}(\dot\beta)\zeta\nu\rho}{(2k_3\zeta + 4k_3)\,\rho^2 + 2k_3\zeta},$$

$$\beta = \begin{cases} \beta_0 + \beta_1(t-t_0) - \dfrac{\zeta\nu\rho\,\mathrm{sgn}(\dot\beta)}{(2k_3\zeta+4k_3)\,\rho^2+2k_3\zeta}\dfrac{(t-t_0)^2}{2}, & (t-t_0) < T; \\ \beta_0 + \beta_1 T - \dfrac{\zeta\nu\rho\,\mathrm{sgn}(\dot\beta)}{(2k_3\zeta+4k_3)\,\rho^2+2k_3\zeta}\dfrac{T^2}{2}, & (t-t_0) \geq T, \end{cases} \tag{2}$$

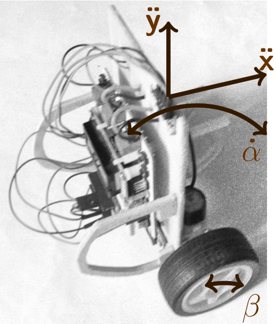
where

$$T = \frac{1}{\beta_1}\frac{\zeta\nu\rho\,\mathrm{sgn}(\dot\beta)}{(2k_3\zeta+4k_3)\,\rho^2+2k_3\zeta}, \quad \{t_0, \beta_0, \beta_1\} \in \mathbb{R}.$$

Theorem

There exists the set of the parameters $\zeta, \rho, k_1, k_2, k_3$ when solution (2) is an attractor as $(t - t_0) < T$.

Observability and sensors



To use a feedback controller for stabilizing of the WIP one should obtain data of the sensors at the moment $t_i = t_{i-1} + dt$.
The gyroscope gives us the angle velocity of the pendulum $\dot\alpha \to a_1$;
The tilt sensor gives us the linear accelerations $(\ddot x, \ddot y) \to a_2$;
The encoder gives us the digital value of the wheel angle $\beta \to b_1 = (\beta_i - \beta_{i-1})/(dt)$.

$$\frac{a_2}{\cos(\alpha)} = -(\cos(\alpha - z)(\frac{\sin(\alpha)\gamma}{\cos(\alpha)\rho} - \frac{a_2\sin(\alpha)}{\cos(\alpha)\rho} + b_2) + b_1^2\sin(\alpha - z))\rho - \frac{2u\rho}{\zeta} + \sin(\alpha), \quad (3)$$

$$(\zeta+2)(\frac{\sin(\alpha)\gamma}{\cos(\alpha)\rho} - \frac{a_2\sin(\alpha)}{\cos(\alpha)\rho} + b_2)\rho \in \begin{cases} f - \nu\mathrm{sgn}(b_1), & b_1 \neq 0; \\ (-\nu, \nu), & b_1 = 0 \cup |f| < \nu; \\ f, & b_1 = 0 \cup |f| \geq \nu; \end{cases} \quad (4)$$

$$f = -\sin(z) - (a_2\cos(\alpha - z) - a_1^2\sin(\alpha - z))\zeta + \frac{2}{\rho}u.$$

Theorem about the observability

Theorem

Let one know the values of the acceleration $(\ddot{x}, \ddot{y})$*, angle velocity of the pendulum* $\dot{\alpha}$ *and angle velocity of the wheel* $\dot{\beta}$*, then the observed dynamical system is solution of the trigonometric equation (3) and the inclusion (5).*

Observed data

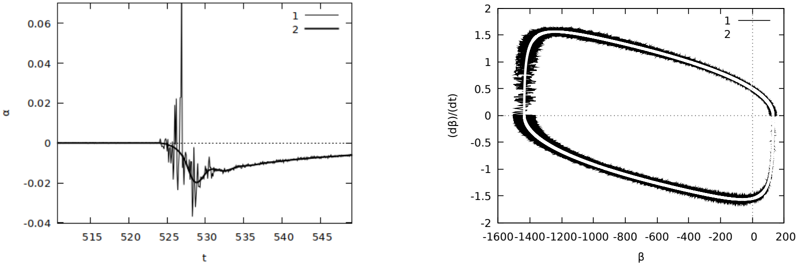
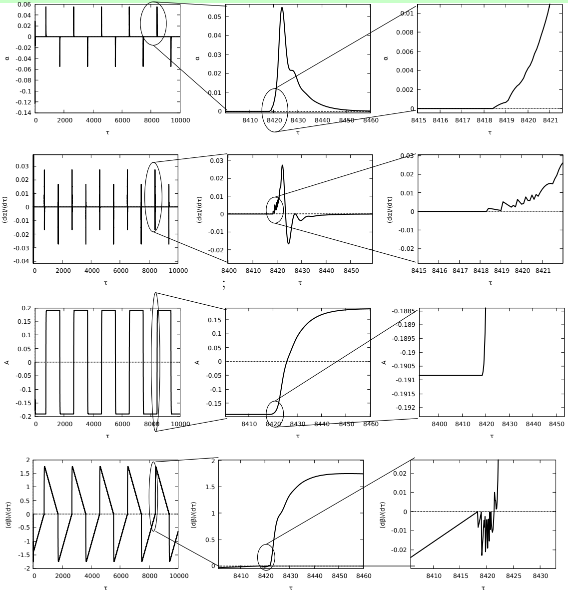


Figure: The observed data on the plane (t, α) and on the phase plane $(\beta, \dot{\beta})$ as $z \equiv 0$ for the WIP on the soft surface. The feedback controller use the tilt sensor and the gyroscope. The parameters of the system are following: $\rho = 0.2$, $\zeta = 10$, $\nu = 0.05$,$\gamma = 1$, the PID coefficients are: $k_1 = 1.7$, $k_2 = 0.2$, $k_3 = 0.02$. The relative errors are uniform distributed data at the interval $(-0.02, 0.02)$. The dynamic system (1) solved at $A \sim 0.2385$, $\alpha = 0.02$, $\dot{\alpha} = 0$, $\beta = 0$, $\dot{\beta} = 0.5$ by Runge-Kutta method of fourth-order method with the step 0.1.

Dynamics on soft surface

Define δ the error of the observed data of $\dot\beta$.
The layer $|\dot\beta| \le \max\{\delta\}$ appears in the stochastic system near the hyperplane $\dot\beta = 0$. In this layer the term $\nu\text{sgn}(\dot\beta)$ takes the random values $\pm\nu$ at $t \in (t_i, t_{i+1})$.
There exists the small neighbourhood $(\Delta_\pm)$ near the unstable lines $(A_\pm, 0, 0, 0, \dot\beta)$, where can be obtained four typical cases:

Trajectories of the WIP



- Let $\dot\beta > 0$, $A < A_+$
  - and $\mathrm{sgn}\breve{\dot\beta} = 1$, then the trajectory is kept in the neighbourhood of the line $(A_+, 0, 0, 0, \dot\beta)$;
  - and $\mathrm{sgn}\tilde{\dot\beta} = -1$, then the trajectory is kept in the neighbourhood of the line $(A_+, 0, 0, 0, \dot\beta)$.
- Let $\dot\beta < 0$, $A > A_-$
  - and $\mathrm{sgn}\breve{\dot\beta} = -1$, then the trajectory is kept in the neighbourhood of the line $(A_-, 0, 0, 0, \dot\beta)$;
  - and $\mathrm{sgn}\breve{\dot\beta} = 1$, then the trajectory is kept in the neighbourhood of the line $(A_-, 0, 0, 0, \dot\beta)$.

The sequence of the changes of the trajectories at the neighbourhoods of the lines $(A_\pm, 0, 0, 0)$ leads to the appearance of the hysteresis loop at the phase plane $(\beta, \dot\beta)$, see figure 2.

Phase portrait under PID-controller

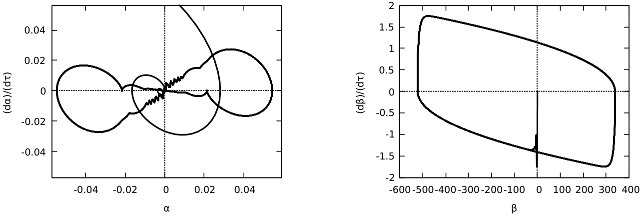


Figure: Here one can see the solution of (1) with the initial conditions $\tau = 0$, $\alpha = -0.1, \dot\alpha = -0.1, \beta = 0, \dot\beta = -0.1$ and under the proportional-integral-derivation controller, where $k_1 = 1.5$, $k_2 = 0.2$, $k_3 = 0.05$. On the left-hand side it is shown the limit cycle on the phase plane $(\alpha, \dot\alpha)$ and on the right-hand side it is shown the limit cycle on the phase plane $(\beta, \dot\beta)$. The parameters of the pendulum are following: $\zeta = 10$, $\rho = 0.2$ and $\nu = 0.1$.

The stochastic data for the PID-controller

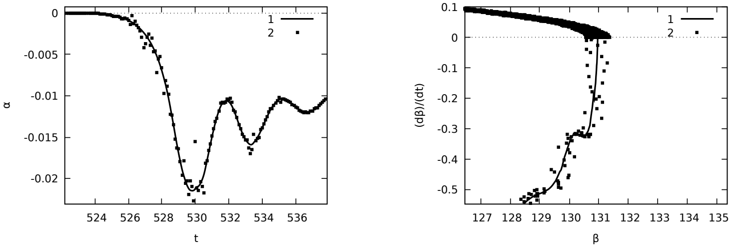


Figure: In this picture one can see the result of the numeric modelling of stochastically controlled WIP at $\xi = 10$, $\rho = 0.2$, $\nu = 0.05$, $\gamma = 1$, $k_1 = 1.7$, $k_2 = 0.2$, $k_3 = 0.02$. On the left picture the plane (t, α) and the right one shows the plane $(\beta, \dot\beta)$. The step of the change of the control torque is 0.1. The line is the solution under the discrete control. The results of the measurements $\ddot{\breve\alpha}_i$, $\dot{\breve\alpha}_i$ and $\dot{\breve\beta}_i$ are modelled by the current values $\ddot\alpha_i$, $\dot\alpha_i$ and $\dot\beta_i$ with the uniform distribution of the relative errors at the interval $(-0.003, 0.003)$. The value of the angle $\breve\alpha_i$ is defined as the observed data. The value $\breve A$ is computed using $\breve\alpha$, $\dot{\breve\alpha}$ integrating by the trapezoidal method. It allows to obtain the control torque u_{i+1} at the interval $t \in (t_i, t_{i+1})$, where $t_{i+1} - t_i = \Delta t$. At $t \in (t_i, t_{i+1})$ the system for the WIP is solved for the constant value the control torque $u = u_{i+1}$ by the Runge-Kutta method of 4-th order and step 0.01.

Average time at $\alpha \equiv 0$

The time between the sequence measurements is equal Δt. Let the trajectory be in the neighbourhood $\Delta_\pm$ of the unstable line. The probability of $\mathrm{sgn}(\delta) = \pm 1$ in primary order as $\Delta_\pm \to 0$ equals $p_\pm \sim 1/2$. The average time for trajectory in this neighbourhood is following:

$$T_0 = \Delta t \sum_{n=1}^{\infty} \frac{n}{2^n} = 2\Delta t.$$

Theorem

The average time spending at Δ-neighbourhood of the unstable lines $(A_\pm, 0, 0, 0, \dot\beta)$ for the stochastic system (1) equals $2\Delta t$, where Δt is the time between the sequenced measurements of the state for the system.

Experimental data

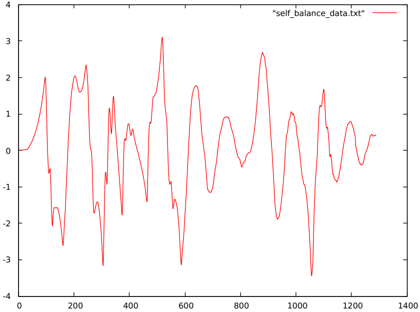
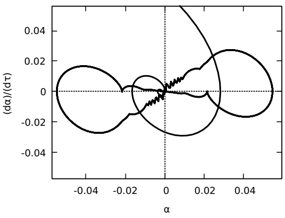
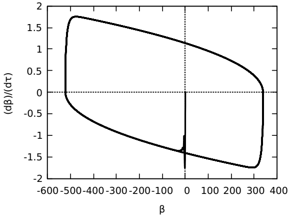


Figure: The experimental data from the wheeled robot with "bad" gear-box under PID-controller.

Conclusions

- If the robot contains the gyroscope, the tilt sensor and the encoder, then the system is observable.
- On the soft surface PID-controller stabilizes of the WIP on the stable limit circle near the upper position of the pendulum.





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