Capture into parametric autoresonance in non-linear oscillator

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Outline

1. Motivation

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where frequency of the perturbation Ω slowly decreases:

Example

$$\Omega(t,\mu) = 2 - \mu^2\lambda^2 t$$

Here λ is additional parameter of perturbation.

Linear approximation

For amplitude of order μ such as $u = \mu v$ one obtains a perturbed Mathieu equation:

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For unperturbed Mathieu equation one can see intervals of resonant frequencies.
The primary resonance takes place near $\Omega = 2$.
At the resonance frequencies solutions exponentially grows. But for nonlinear equation like equation for pendulum such growth can't be for long period. The solution grows up $O(\sqrt{\mu})$.

$u'' + (1 + 4\mu\cos(\Omega t))\sin(u) = 0$

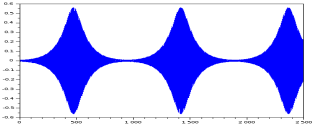

Рис.: Typical trajectory for nonlinear parametric resonance $\Omega = 2$. Perturbation parameter $\mu = 0.01$, amplitude of oscillations 0.5

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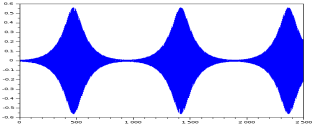

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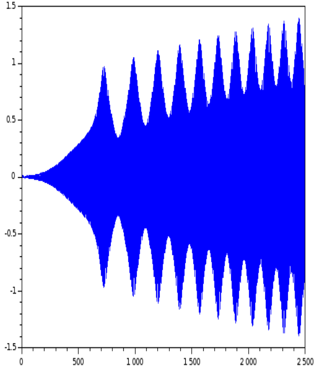

Рис.: Trajectory of parametric autoresonance driven pendulum $\Omega = 2 - \mu^2\lambda^2 t$. Amplitude of oscillations grows up to separatrix of of pendulum (1.5).

Parametric autoresonance

To find asymptotic solution of order $\sqrt{\mu}$ which is suitable over ling time we assume:

$$u = \sqrt{\mu} A(\tau) \exp(it) + \mu^{3/2} U(t, \mu) + c.c.,$$

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- first term defines modulated oscillations with slow varying amplitude $A(\tau)$,
- second term defines a reminder,
- $c.c.$ defines complex conjugated terms.

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Let us substitute this formula into equation for perturbed pendulum. As a result we obtain in order of $\mu^{3/2}$:

$$U'' + U \sim -2iA'e^{it} + \frac{1}{2}|A|^2Ae^{it} - 2\overline{A}e^{it+i\omega\tau} + \frac{1}{6}A^3e^{3it} + 2Ae^{3it+i\omega\tau} + c.c..$$

Here one can see three resonant terms

- $-2iA'e^{it}$ this term appears due to assumption about slow varying amplitude of oscillations.

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An exception of these terms from equation for $U(t, tau, \mu)$ yields:

$$-2iA' + \frac{1}{2}|A|^2A - 2\overline{A}e^{i\omega\tau} = 0.$$

Equation for primary resonance

Let us define $A = 2\psi e^{i\omega\tau/2}$ and simplest chirp-rate $\omega = -\lambda^2\tau$, then we obtain:

$$i\psi' + (\lambda^2\tau - |\psi|^2)\psi + \overline{\psi} = 0,$$

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For argument and modulus of complex-valued function $\psi(\tau) = R(\tau)\exp(i\phi(\tau))$ one obtains a system:

$$\phi' + R^2 - \lambda^2\tau - \cos(2\phi) = 0, \quad R' - R\sin(2\phi) = 0.$$

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Most elegant form of equation for the parametric autoresonance, which looks like as one equation of second order for $\phi = \varphi/2$:

$$\varphi'' + 4\lambda^2\tau\sin(\varphi) + 2\sin(2\varphi) - 2\lambda^2 = 0. \tag{2}$$

References

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Our problem is a scattering over such resonances for nonlinear oscillators.

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- O.M. Kiselev and S.G. Glebov. The capture into parametric autoresonance. *Nonlinear Dynamics*, 2007.

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Scattering for large solutions. Numerics

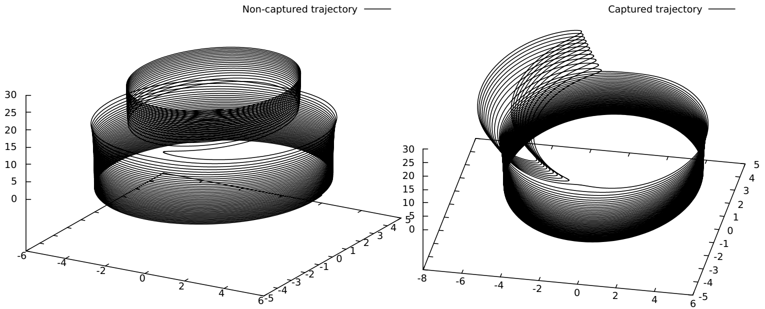

Рис.: On the left-hand side one can see a solution of (8) with initial condition $\psi = 5\exp(0.15i)$ at $\tau = 0$. The trajectory turns at $\tau \sim 20$. On the right-hand side one can see a solution of (8) with initial condition $\psi = 5\exp(0.19i)$ at $\tau = 0$. The graph shows how this trajectory is captured at $\tau \sim 20$. Both trajectories are constructed by Runge-Kutta method of 4-th order with step 0.001.

Equation for large value

A goal of this work is studying of the capture into parametric resonance of large amplitude solutions of (8) as $\tau \to \infty$. It is convenient to investigate such solutions of (8) using a special depending on an inverse value of small parameter:

$$\psi = \varepsilon^{-1}\Psi(\tau, \varepsilon), \quad 0 < \varepsilon \ll 1.$$

Here ε^{-1} is a parameter of solution, which defines an amplitude of oscillations of ψ.

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After substitution (8) one gets:

$$i\varepsilon^2\Psi' + (\lambda^2\varepsilon^2\tau - |\Psi|^2)\Psi + \varepsilon^2\overline{\Psi} = 0. \tag{3}$$

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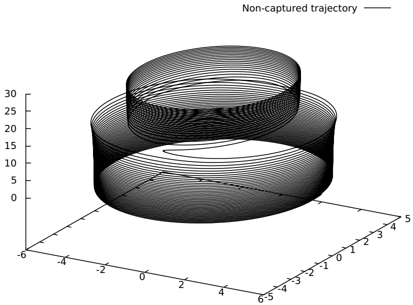
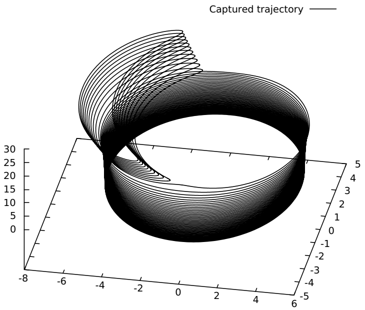
- μ defines a perturbation of nonlinear oscillator;
- ε^{-1} defines an amplitude of the solution for the oscillator
- The order of amplitudes of oscillations, which are considered here, is *intermediate small:* $A(\tau) = O\left(\frac{\sqrt{\mu}}{\varepsilon}\right)$ and $\frac{\sqrt{\mu}}{\varepsilon} \ll 1$

$$i\varepsilon^2\Psi' + (\lambda^2\varepsilon^2\tau - |\Psi|^2)\Psi + \varepsilon^2\overline{\Psi} = 0.$$



- A coefficient $(\lambda^2\varepsilon^2\tau - |\Psi|^2)$ in the equation can change a sign when τ is large $\tau = O(\varepsilon^{-2})$.
- Such changing leads to turn of the trajectory, this can be seen on the left-hand side of figure.

$$i\varepsilon^2\Psi' + (\lambda^2\varepsilon^2\tau - |\Psi|^2)\Psi + \varepsilon^2\overline{\Psi} = 0.$$



- In rare cases the change of sign for $(\lambda^2\varepsilon^2\tau - |\Psi|^2)$ leads to capture into the parametric autoresonance.

Really a parameter ε defines a value of modulus of ψ. Therefore ε is a parameter of solution for (8). Without a loss of generality one can assume that the expression $(\lambda^2\varepsilon^2\tau - |\Psi|^2)$ equals to zero at $\tau_* = (\varepsilon\lambda)^{-2}$.

Equation for phase function

To study the capture we present the equation as an equation for angle variable. Then we get:

$$\varphi'' + 4(\varepsilon^{-2} + \lambda^2(\tau - \tau_*))\sin(\varphi) - 2\lambda^2 + 2\sin(2\varphi) = 0.$$

Let us consider new scale of the independent variable: $\theta = 2(\tau - \tau_*)/\varepsilon$.

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Let us consider new scale of the independent variable: $\theta = 2(\tau - \tau_*)/\varepsilon$. As a result we obtain explicit form for the perturbation with respect to ε:

$$\varphi_{\theta\theta} + \sin(\varphi) - \varepsilon^2\frac{\lambda^2}{2} + \varepsilon^2\frac{\sin(2\varphi)}{2} = -\varepsilon^3\theta\frac{\lambda^2}{2}\sin(\varphi). \qquad (4)$$

The left-hand side of equation is integrable and only perturbation on the right-hand side introduces an obstacle for integrability.

Unperturbed equation

$$\varphi_{\theta\theta} + \sin(\varphi) - \varepsilon^2 \frac{\lambda^2}{2} + \varepsilon^2 \frac{\sin(2\varphi)}{2} = 0. \tag{5}$$

When $\varepsilon = 0$ the equation (5) equals to the pendulum equation. Stationary solutions of (5) are close to equilibriums of pendulum.

$$\sin(\varphi_k) - \varepsilon^2 \frac{\lambda^2}{2} + \varepsilon^2 \frac{\sin(\varphi_k)}{2} = 0, \quad k \in \mathbf{Z}.$$

For these equilibriums one can obtain an asymptotic formula:

$$\varphi_k \sim \pi k + \varepsilon^2 (-1)^k \frac{\lambda^2}{2} - \varepsilon^4 \frac{\lambda^2}{2} + O(\varepsilon^6), \quad k \in \mathbf{Z},$$

where $\varphi_{2n}, n \in \mathbf{Z}$ are centers and $\varphi_{2n+1}, n \in \mathbf{Z}$ are saddles. Trajectories of (5) are shown below.

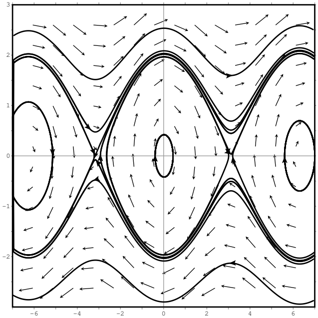

Рис.: Phase portrait for (5), at $\varepsilon = 0.2$, $\lambda = 1$. One can see periodic solutions near a center, homoclinics which begin and finishing near saddles and twisted trajectories, which pass between left point of the homoclinics and nearest saddle.

Slow varying equilibriums.

For perturbed equation one can construct an algebraic asymptotic expansions, which correspond to equilibriums of non-perturbed equation (5). Such asymptotic expansions can be obtained by using the regular perturbation theory.

$$\varphi_k \sim \pi k - \varepsilon^2(-1)^k\frac{\lambda^2}{2} + \varepsilon^4\frac{\lambda^2}{2} + \varepsilon^5(-1)^k\theta\frac{\lambda^4}{4}O(\varepsilon^6), \quad k \in \mathbf{Z}. \qquad (6)$$

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This expansion can be used as $|\varepsilon\theta| \ll 1$. For large values of θ one should use another form of equation (4) where new independent variable is defined as $\sigma = \varepsilon^3\theta$. On such way one obtains:

$$\varepsilon^6\varphi_{\sigma\sigma} + (1 + \frac{\lambda^2}{2}\sigma)\sin(\varphi) - \varepsilon^2\frac{\lambda^2}{2} + \varepsilon^2\frac{1}{2}\sin(2\varphi) = 0.$$

To study solutions of this equation for large values of σ one should make changing of independent variable:

$$S = \frac{1}{\varepsilon^3}\int^{\sigma}\sqrt{1+\lambda^2\sigma/2}d\sigma.$$

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Then equation has a form:

$$\frac{d^2\varphi}{dS^2} + \varepsilon^3\frac{\sqrt{2}\lambda^2}{(\sqrt{2+\lambda^2\sigma})^3}\frac{d\varphi}{dS} + \sin(\varphi) + \varepsilon^2\frac{\sin(2\varphi)}{2(1+\sigma/2)} - \varepsilon^2\frac{\lambda^2}{2(1+\sigma/2)} = 0.$$

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For $S = \frac{4}{3\lambda^2\varepsilon^3}(\sqrt{1+\lambda^2\sigma/2})^3$ we get

$$\frac{d^2\varphi}{dS^2} + \sin(\varphi) + \frac{\sqrt[3]{2}\sin(2\varphi)}{\lambda^{4/3}(3S)^{2/3}} - \frac{\sqrt[3]{2\lambda^2}}{(3S)^{2/3}} + \frac{2}{3S}\frac{d\varphi}{dS} = 0. \qquad (7)$$

Slow varying solutions of $\frac{d^2\varphi}{dS^2} + \sin(\varphi) + \frac{\sqrt[3]{2}\sin(2\varphi)}{\lambda^{4/3}(3S)^{2/3}} - \frac{\sqrt[3]{2\lambda^2}}{(3S)^{2/3}} + \frac{2}{3S}\frac{d\varphi}{dS} = 0$

For slowly varying equilibriums of equation for perturbed pendulum one can obtain asymptotic formula:

$$\varphi_k \sim \pi k + \sqrt[3]{\frac{2\lambda^2}{9S^2}}(-1)^k - \frac{2}{3S}\sqrt[3]{\frac{4}{3\lambda^2 S}} + O(S^{-2}). \tag{8}$$

Kuznetsov's theorem about algebraic asymptotics (A. N. Kuznetsov, *Funct. Anal. Appl.*, 6(2):119–127, 1972.) yields:

Theorem

There exist solutions of (4) with asymptotic expansions (8).

Neighborhood of equilibriums

Let us study solutions near ϕ_k, $k = 2n, n \in \mathbf{Z}$ for perturbed equation.
Define by an action, or the same, square, which are enveloped of the curve for one oscillation:

$$I = \int_{\mathcal{L}} \varphi' d\varphi,$$

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$\mathcal{L}$ is a curve (φ, φ'), which is defined by equation:

$$E = \frac{(\varphi')^2}{2} - \cos(\varphi) - \varepsilon^2 \frac{\lambda^2}{2}\varphi - \varepsilon^2 \frac{\cos(2\varphi)}{4}, \quad E < 1.$$

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The evolution of E under the perturbation can be calculated:

$$\frac{dE}{d\theta} = -\varepsilon^3 \frac{\lambda^2}{2} \theta \sin(\varphi)\varphi'. \qquad (9)$$

During one oscillation parameter E changes on a value:

$$\begin{aligned}\delta E \quad &= -\varepsilon^3\frac{\lambda^2}{2}\int_\theta^{\theta+\Theta}\theta\sin(\varphi)\varphi' d\theta \sim \varepsilon^3\frac{\lambda^2}{2}\int_\theta^{\theta+\Theta}\theta\varphi''\varphi' d\theta = \\ &\varepsilon^3\frac{\lambda^2}{4}\theta(\varphi')^2\Big|_\theta^{\theta+\Theta} - \varepsilon^3\frac{\lambda^2}{2}\int_\theta^{\theta+\Theta}(\varphi')^2 d\theta = -\varepsilon^3\frac{\lambda^2}{4}I.\end{aligned}$$

During one oscillation parameter E changes on a value:

$$\delta E = -\varepsilon^3 \frac{\lambda^2}{2} \int_\theta^{\theta+\Theta} \theta \sin(\varphi)\varphi' d\theta \sim \varepsilon^3 \frac{\lambda^2}{2} \int_\theta^{\theta+\Theta} \theta \varphi'' \varphi' d\theta = \varepsilon^3 \frac{\lambda^2}{4} \theta(\varphi')^2\Big|_\theta^{\theta+\Theta} - \varepsilon^3 \frac{\lambda^2}{2} \int_\theta^{\theta+\Theta} (\varphi')^2 d\theta = -\varepsilon^3 \frac{\lambda^2}{4} I.$$

The changing of the action variable are the same:

$$\delta I = -\varepsilon^3 \frac{\lambda^2}{4} I. \tag{10}$$

During one oscillation parameter E changes on a value:

$$\begin{aligned}\delta E \quad &= -\varepsilon^3\frac{\lambda^2}{2}\int_\theta^{\theta+\Theta}\theta\sin(\varphi)\varphi' d\theta \sim \varepsilon^3\frac{\lambda^2}{2}\int_\theta^{\theta+\Theta}\theta\varphi''\varphi' d\theta = \\ &\varepsilon^3\frac{\lambda^2}{4}\theta(\varphi')^2\Big|_\theta^{\theta+\Theta} - \varepsilon^3\frac{\lambda^2}{2}\int_\theta^{\theta+\Theta}(\varphi')^2 d\theta = -\varepsilon^3\frac{\lambda^2}{4}I.\end{aligned}$$

The changing of the action variable are the same:

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This means, that the action or the same a square inside the trajectory over one oscillation decreases. A projection of the curve on a plane (φ, φ') is smooth. Therefore the oscillating solution of perturbed equation tends to φ_{2n}. It means φ_{2n} is stable focus.

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Theorem

The solution φ_{2n} is a stable focus for (4).

Neighborhood of the focus

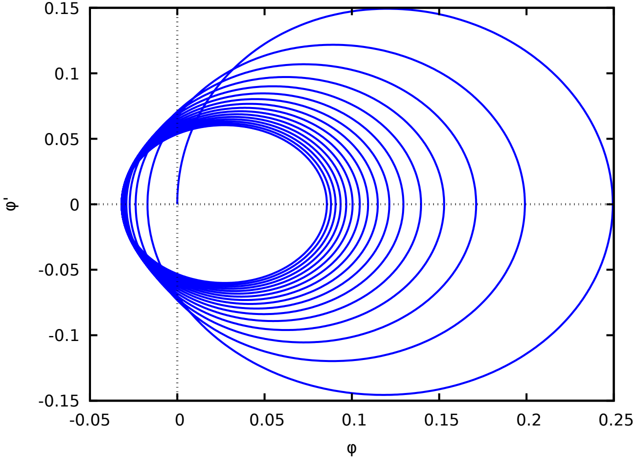

Рис.: The initial point of the trajectory is (0.0). The area of the projection decreases.

The evolution of E for perturbed equation is defined by following equation:

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$$\Delta = -\int_{-\infty}^{\infty}\varepsilon^3\frac{\lambda^2}{2}\theta\sin(\varphi)\varphi' d\theta.$$

A value of this integral defines a gap between two separatrix of perturbed equation near the saddle (V.K. Mel'nikov. *Trudy Moskov. Mat. Obshch.*, 12:3—-52, 1963).

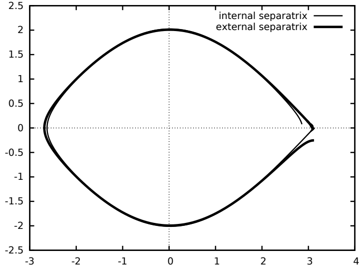

Рис.: The splitting of separatrix for equation (4). The external curve make a loop and tends to the saddle. The internal curve goes from the saddle and tends to the focus.

Asymptotic value of the gap is:

$$\Delta \sim \varepsilon^3 \frac{\lambda^2}{2} \int_{-\infty}^{\infty} \theta \varphi'' \varphi' d\theta \sim \varepsilon^3 \frac{\lambda^2}{2} \int_{\mathcal{L}} \varphi' d\varphi.$$

Here $\mathcal{L}$ is the separatrix loop of equation (5). As a result

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Trajectories which go through the gap into the loop of non-perturbed equation remain into this loop. Due to the equation (10) their action decreases and the trajectories tend to focus φ_{2n}. Such trajectories are captured into the autoresonance.

Theorem

The trajectories passed through the gap between separatrices

$$\varphi|_{\theta=\theta_k} = \varphi_{2k+1}, \quad -2\varepsilon\sqrt{2\varepsilon}\lambda < \frac{d\varphi}{d\theta}|_{\theta=\theta_k} < 0, \quad \lambda > 0. \qquad (11)$$

as $\theta_k < \theta < \mathcal{O}(\mu^{-3})$ *will oscillate near the focus* φ_{2k}.

To find trajectories which will go through the gap between separatrices for perturbed equation and will be captured into the autoresonance we consider a Cauchy problem for equation (5) for a family of solutions with thin profile into the gap of separatrices (11).

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Let us calculate the evolution of E for perturbed equation. Direct substitution gives:

$$\frac{dE}{d\theta} = \varepsilon^3\lambda^2\theta\sin(\varphi)\varphi_\theta.$$

Values

$$E_k^-|_{\varphi=\varphi_k} = -\cos(\varphi_k) - \varepsilon^2\lambda^2\varphi_k - \varepsilon^2\frac{\cos(2\varphi_k)}{4}$$

and

$$E_k^+|_{\theta=\theta_k} = -\cos(\varphi_k) - \varepsilon^2\lambda^2\varphi_k - \varepsilon^2\frac{\cos(2\varphi_k)}{4} + \varepsilon^3\frac{\lambda^2}{4}S_{\mathcal{L}}$$

define a projection of captured area on the complex plane Ψ.

The projection looks like two spirals which twisted with frequency $\mathcal{O}(1/\varepsilon^{-3})$ at distance $O(\varepsilon^{-1})$ from $\Psi = 0$.

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Let us estimate a square of this areas during one twist:

$$S = 2\pi\varepsilon^{-1}\Delta R \sim 4\pi\varepsilon^{3/2}\sqrt{2}\lambda.$$

Hence the measure of captured trajectories from $R = R_0$ up to $R \to \infty$ equals to integral from $1/R_0$ up to infinity.

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Theorem

Beginning at some large $\tau = \tau_0$ up to infinity the measure of captured trajectories has a following asymptotics

$$M \sim \frac{16\pi\lambda^2}{\tau_0}, \quad \tau_0 \to \infty.$$

The separatrices which bound projections of captured trajectories we can define as solutions of following Cauchy problems:

$$
\begin{aligned}
\frac{dE}{d\theta} &= \varepsilon^3\lambda^2\theta\sin(\varphi)\phi_\theta,\\
\left(\frac{d\varphi}{d\theta}\right)^2 &= 2E + 2\cos(\varphi) + 2\varepsilon^2\lambda^2\phi + \varepsilon^2\frac{\cos(2\varphi)}{2},\\
E_k^-|_{\theta=\theta_k} &= -\cos(\varphi_k) - \varepsilon^2\lambda^2\varphi_k - \varepsilon^2\frac{\cos(2\varphi_k)}{4},\\
E_k^+|_{\theta=\theta_k} &= -\cos(\varphi_k) - \varepsilon^2\lambda^2\varphi_k - \varepsilon^2\frac{\cos(2\varphi_k)}{4} + \varepsilon^3\frac{\lambda^2}{4}S_{\mathcal{L}},\\
\varphi|_{\theta=\theta_k} &= \varphi_{2k+1}.
\end{aligned}
\tag{12}
$$

The separatrices correspond by different initial values for parameter $E = E_k^+$ and E_k^-.

One can consider system of equations (15) for E and φ as along equation for E.

$$\frac{dE}{d\varphi} = \varepsilon^3\lambda^2\sin(\varphi)\int_{\varphi_k}^{\varphi}\frac{d\varphi}{\sqrt{2E + 2\cos(\varphi) + 2\varepsilon^2\lambda^2\varphi + \varepsilon^2\frac{\cos(2\varphi)}{2}}}, \tag{13}$$

$$E|_{\varphi_{2k+1}} = E_k^{\pm}. \tag{14}$$

The Cauchy problem (14) defines the asymptotic behavior of separatrices which bound the captured trajectories.

Main results

- The captured trajectories are defined by Cauchy problem

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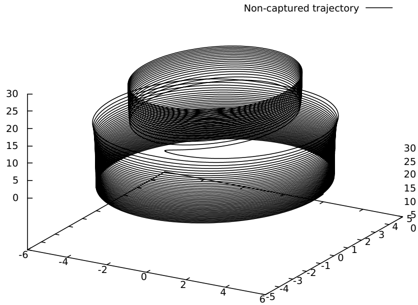
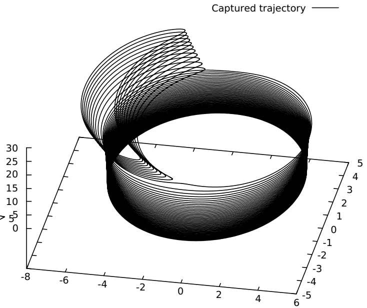
- Measure of trajectories which are captured into parametric autoresonace for large time is bounded.





O. M. Kiselev, Asymptotic behaviour of measure for captured trajectories into parametric autoresonance, 2016, 13 pp., arXiv:1612.08426