Evolution of basins of attraction in perturbed Painlevé-2 equation

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Perturbed Painlevé-2

$$u'' = -2u^3 + xu - \varepsilon f(u, u', x), \quad 0 < \varepsilon \ll 1. \tag{1}$$

- Dynamical bifurcations.
- Passages through separatrices.
- Captures into resonances.

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The soft loss of a stability

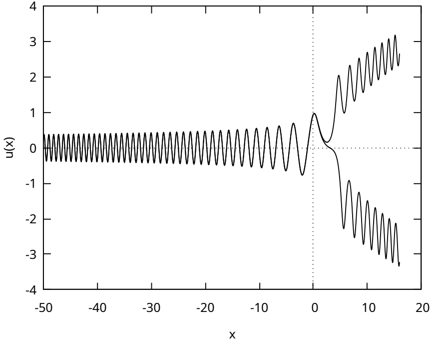


Figure: The image shows two numerical solutions of the unperturbed Painlevé-2 equation with initial conditions taken near the bifurcation boundary. For $x < 0$, the curves almost coincide; for $x > 0$, they diverge as a result of soft loss of stability.

Asymptotic properties of solutions

$$u \sim \frac{\alpha}{\sqrt[4]{-x}} \sin\left(\frac{2}{3}(-x)^{3/2} + \frac{3}{4}\alpha^2 \log(-x) + \phi\right), \quad x \to -\infty. \tag{2}$$

The solution parameters are arbitrary constants α and ϕ.
In the right part of the figure 1 solutions for $x \to \infty$ oscillate in the neighbourhood of branches of the function $\pm\sqrt{x/2}$.

The shadow bifurcation border

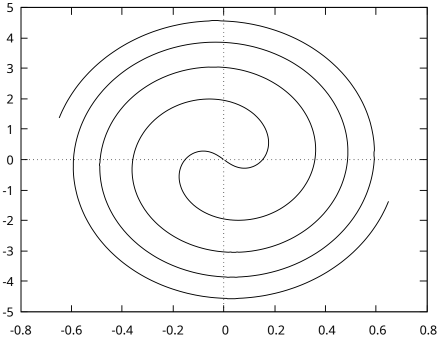
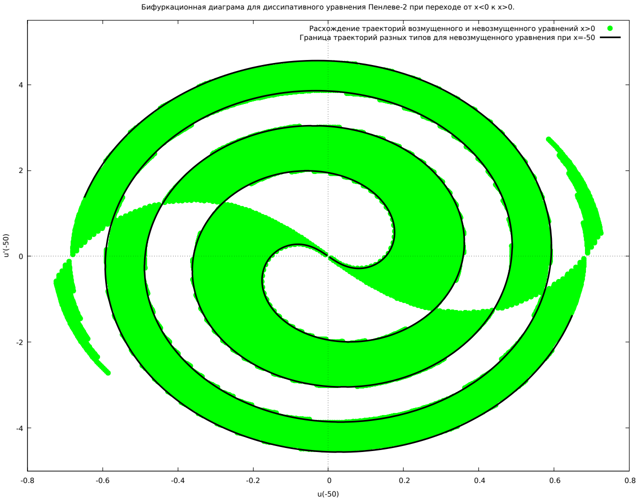


Figure: Here is shown the bifurcation boundary for the Painlevé-2 transcendent in the cross section of the phase space at $x = -50$.

$$\frac{3}{2}\alpha^2 \log(2) - \frac{\pi}{4} - \arg\left(\Gamma\left(\frac{i\alpha^2}{2}\right)\right) = 0. \qquad (3)$$

Motivation


Бифуркационная диаграма для диссипативного уравнения Пенлеве-2 при переходе от x<0 к x>0.
Расхождение траекторий возмущенного и невозмущенного уравнений x>0
Граница траекторий разных типов для невозмущенного уравнения при x=-50
u'(-50)
u(-50)
4
2
0
-2
-4
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8

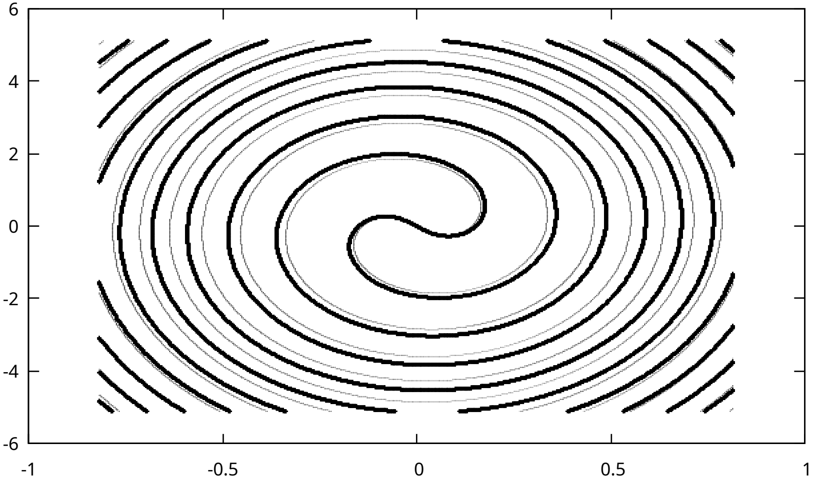
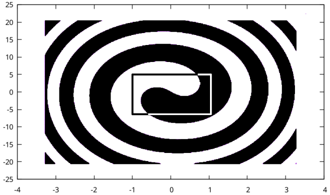


Figure: Here one can see the results of calculations of 2048x4096 trajectories by the Runge-Kutta method of the 4th order. The bifurcation boundary in the section of the phase space (u, u', x) is given for $x = -50$. The bold curve corresponds to the Painlevé-2 equation, the thin curve corresponds to the perturbed equation (1) with perturbation $f = u(u')^2$ for $\varepsilon = 0.1$.



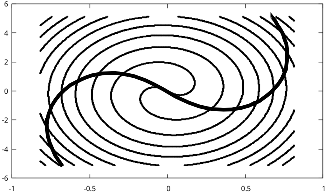


Figure: Here the results of calculations of 2048x4096 trajectories by the Runge-Kutta method of the 4th order are shown. The set of trajectories in the cross section of the phase space (u, u', x) at $x = -50$ for the equation (1) with perturbation $f = u'$ at $\varepsilon = 0.1$ is shown on the left. The dark part is the set of starting points of trajectories that, when passing through $x = 0$, fall into the neighbourhood of $\sqrt{x/2}$. The bifurcation boundary of the set of trajectories is shown from the rectangle selected in the left image. In the right picture, the thin curves correspond to the Painlevé-2 equation, while the thick one corresponds to the perturbed equation (1) with the perturbation $f = u'$ at $\varepsilon = 0.1$.

Asymptotic substitutions

The asymptotics for the parameter ε will be constructed as:

$$u(x,\varepsilon) \sim \sum_{k=0}^{\infty} \varepsilon^k u_k(\xi/\varepsilon, \alpha, \phi),$$

$$\alpha \sim \sum_{k=0}^{\infty} \varepsilon^k \alpha_k(\xi), \quad \phi \sim \sum_{k=0}^{\infty} \varepsilon^k \phi_k(\xi). \tag{4}$$

It yields:

$$u' \equiv \frac{du_k}{dx} = \varepsilon \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \alpha} \varepsilon \frac{\partial \alpha}{\partial \xi} + \frac{\partial u}{\partial \phi} \varepsilon \frac{\partial \phi}{\partial \xi} = \varepsilon \dot{u} + \frac{\partial u}{\partial \alpha} \varepsilon \dot{\alpha} + \frac{\partial u}{\partial \phi} \varepsilon \dot{\phi}.$$

The main condition for representing corrections in the formula (4) is uniform boundedness in ε for $\xi < 0$.

The primary term

An asymptotics of the primary term of the perturbed Painlevé transcendent:

$$u_0(\xi/\varepsilon, \alpha, \phi) \sim \frac{\alpha\sqrt[4]{\varepsilon}}{\sqrt[4]{-\xi}}\sin(s) + \frac{\varepsilon^{7/4}}{(-\xi)^{7/4}}\left(-\frac{3}{8}\alpha^3\sin(s) + \frac{102\alpha^5 - 20\alpha}{192}\cos(s) - \frac{1}{16}\alpha^3\sin(3s)\right) + O\left(\left(\frac{\varepsilon}{-\xi}\right)^{13/4}\right),$$

where:

$$s = \frac{2}{3}(-\xi/\varepsilon)^{3/2} + \frac{3}{4}\int^{\xi/\varepsilon}\alpha^2(\zeta, \varepsilon)\frac{d\zeta}{\zeta} + \phi(\xi, \varepsilon).$$

The first correction

$$\varepsilon^2 \ddot{u}_1 = -6u_0^2 u_1 + x u_1 - f(u_0, \varepsilon \dot{u}_0, \xi/\varepsilon) - 2\varepsilon \dot{\alpha}_0 \partial_\alpha \dot{u}_0 - \varepsilon \dot{\phi}_0 \partial_\phi \dot{u}_0.$$

Two solutions of linearized Painlevé-2 are:

$$u_\alpha \sim \frac{\sqrt[4]{\varepsilon}}{\sqrt[4]{-\xi}} \sin\left(\frac{2}{3}(-\xi/\varepsilon)^{3/2} + \frac{3}{4}\int^{\xi/\varepsilon} \alpha^2(\zeta)\frac{d\zeta}{\zeta} + \phi\right), \quad \xi < 0, \quad \varepsilon$$

$$u_\phi \sim \frac{\sqrt[4]{\varepsilon}}{\sqrt[4]{-\xi}} \cos\left(\frac{2}{3}(-\xi/\varepsilon)^{3/2} + \frac{3}{4}\int^{\xi/\varepsilon} \alpha^2(\zeta)\frac{d\zeta}{\zeta} + \phi\right), \quad \xi < 0, \quad \varepsilon$$

The solution of the equation for the first correction term can be represented using the formula:

$$u_1 = u_\alpha \int_0^{\xi/\varepsilon} (f(u_0, \varepsilon \dot{u}_0, y) - 2\dot{\alpha}_0 \partial_\alpha \dot{u}_0 - 2\dot{\phi}_0 \partial_\phi \dot{u}) u_\phi(y) dy - u_\phi \int_0^{\xi/\varepsilon} (f(u_0, \varepsilon \dot{u}_0, y) - 2\dot{\alpha}_0 \partial_\alpha \dot{u}_0 - 2\dot{\phi}_0 \partial_\phi \dot{u}) u_\alpha(y) dy. \quad (5)$$

A condition for discard the linear growth in the first correction term can be obtained by averaging:

$$\begin{aligned} \xi \dot{\alpha}_0 &= \varepsilon \int_0^{\xi/\varepsilon} f(u_0, \varepsilon \dot{u}_0, y) u_\phi(y) dy, \\ \xi \dot{\phi}_0 &= -\varepsilon \int_0^{\xi/\varepsilon} f(u_0, \varepsilon \dot{u}_0, y) u_\alpha(y) dy. \end{aligned}$$

Proposition

The perturbations in the form:

$$f(u, u', x) = \sum_{4k_1+k_3\leq k_2}^{N} a_{k_1k_2k_3} t^{k_1} u^{k_2} (u')^{k_3},$$

where $(k_1+1), (k_2+1), (k_3+1) \in \mathbb{N}$, $k_1+k_2+k_3 \leq N$, $N \in \mathbb{N}$, *yield the equations for the correction terms in (4) uniformly with respect* ε *and* ξ.

If the condition 1 is true, then the intergands in (5) have the order $O(1)$, as $u_0 \sim \sqrt[4]{\varepsilon}$, and $\varepsilon \dot{u}_0 = O(1/\sqrt[4]{\varepsilon})$), and $\xi/\varepsilon = O(\varepsilon^{-1})$. In the integrand, u_ϕ and u_α are used to explicitly calculate the corrections. These functions have the order $\sqrt[4]{\varepsilon}$. Then for the right side of the form $t^{k_1} u^{k_2} (u')^{k_3}$ we get the order of the expression $\varepsilon^{k_1+k_3/4-k_2/4-1/4}$. The requirement of the boundedness of the integrand for u_1:

$$4k_1 + k_3 = k_2 + 1.$$

This reasoning also applies to an arbitrary-order correction. The 1 statement is proved.

A remainder of the asymptotic series

Let us construct a segment of the formal series of perturbation theory:

$$U_N(\xi,\varepsilon)=\sum_{k=0}^{N}\varepsilon^k u_k(\xi/\varepsilon,\xi).$$

Then for the remainder of the asymptotic $\varepsilon^N U = u(x,\varepsilon)-U_N(\xi,\varepsilon)$ one obtains the equation:

$$\frac{d^2U}{dx^2}=xU-6u_0^2U+\varepsilon F(U,U_N,U',U_N',x).$$

We transform this equation to a system of first-order equations:

$$U'=V,\quad V'=xU-6u_0^2U+\varepsilon F(U,V,U_N,U_N',x).$$

System of equations for the remainder

The linear part of this system has a fundamental solution:

$$R(x) = \begin{bmatrix} u_\alpha & u_\phi \\ u'_\alpha & u'_\phi \end{bmatrix}.$$

We will search for the vector $[U, V]^T = R[X_1, X_2]^T$. For $X = [X_1, X_2]$ we get:

$$X' = \varepsilon R^{-1}(x)[0, F((RX)_1, (RX)_2, U_N, U'_N, x)]^T,$$

$$R^{-1}(x) = \begin{bmatrix} u'_\phi & -u_\phi \\ -u'_\alpha & u_\alpha \end{bmatrix}.$$

It is convenient to rewrite this system of equations in the form:

$$\begin{bmatrix} X'_1 \\ X'_2 \end{bmatrix} = \varepsilon \begin{bmatrix} -u_\phi F(u_\alpha X_1 + u_\phi X_2, u'_\alpha X_1 + u'_\phi X_2, U_N, U'_N, x) \\ u_\alpha F(u_\alpha X_1 + u_\phi X_2, u'_\alpha X_1 + u'_\phi X_2, U_N, U'_N, x) \end{bmatrix}. \quad (6)$$

Then for the remainder of the asymptotic $\varepsilon^N U = u(x,\varepsilon) - U_N(\xi,\varepsilon)$ one obtains the equation:

$$\frac{d^2U}{dx^2} = xU - 6u_0^2U + \varepsilon F(U, U_N, U', U'_N, x).$$

Proposition

If the perturbation f satisfies conditions 1, then for $\forall N \in \mathbb{N}$ there is $\xi_0 \in \mathbb{R}$, $\xi_0 < 0$, and

$$u(x,\varepsilon) = U_{N-1} + O(\varepsilon^N), \quad x \in (\xi_0/\varepsilon, 0), \quad \varepsilon \to 0.$$

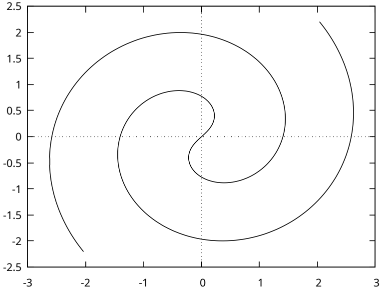


Figure: Bifurcation boundary for the parameters of the Painlevé-2 transcendent in the polar coordinate system, here r is the distance from the coordinate axis, $((3/2)r^2 \log(2) - \pi/4 - \arg(\Gamma(ir^2/2)$ is the angle relative to the abscissa axis.

In the theory of Painlevé transcendent, it is known that for $x \to \infty$ solutions, two families can be divided according to the asymptotic behaviour. These families and their relation to monodromy data were established in the already mentioned

A small dissipation

$$u'' = -2u^3 + xu - \varepsilon u'. \qquad (9)$$

Here:

$$\dot{\alpha}_0 \sim -\frac{1}{2}\alpha_0$$

and

$$\phi_0' \sim 0.$$

The asymptotic behaviour of the primary term of the Painlevé-2 transcendent with small dissipation has the form:

$$u_0 \sim a\frac{e^{-\varepsilon x/2}}{\sqrt[4]{-x}} \sin\left(\frac{2}{3}(-x)^{3/2} + \frac{3a^2}{4}\int^x \frac{e^{-\varepsilon z}}{z} dz + p\right), \quad x \to -\infty.$$

Here a and p are solution parameters.

Asymptotics and numeric results

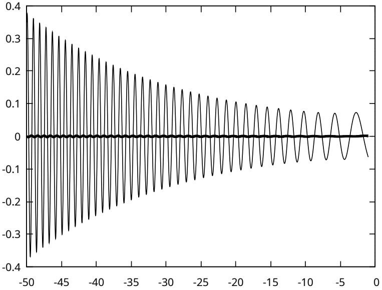


Figure: The numerical solution of the equation (9) on the figure practically coincides with the constructed asymptotic solution. Therefore, the figure shows the numerical solution, which is a thin line, and the difference between the numerical solution and the constructed asymptotic solution, which is a bold line near the abscissa axis.

Numeric and asymptotic boundaries

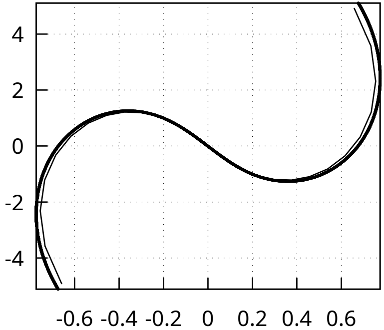


Figure: Here one can see the cross-section of the u, u' bifurcation boundary at $x = -50$ for solutions of the perturbed Painlevé-2 equation with small dissipation (9) at $\varepsilon = 0.1$. the bold line is the boundary obtained numerically from 2048x4096 trajectories, with the beginning at $x = -50$. A thin line is a boundary calculated from perturbation theory.

A nonlinear perturbation

Here we consider another example of the perturbed Painlevé-2 equation:

$$u'' = -2u^3 + xu - \varepsilon (u')^2 u. \tag{10}$$

For this equation, the parameter α is a constant

$$\dot{\alpha}_0 \sim 0$$

and the equation for modulated ϕ follows:

$$\dot{\phi} \sim -\frac{1}{8}\alpha_0^3.$$

That is, the perturbation leads to a shift:

$$u_0 \sim \frac{\alpha_0}{\sqrt[4]{-x}} \sin\left(\frac{2}{3}(-x)^{3/2} + \frac{3\alpha_0^2}{4}\log(-x) - \frac{1}{8}\alpha^3 \varepsilon x + p\right). \tag{11}$$

Here α and p are solution parameters.

Numeric and asymptotic solutions

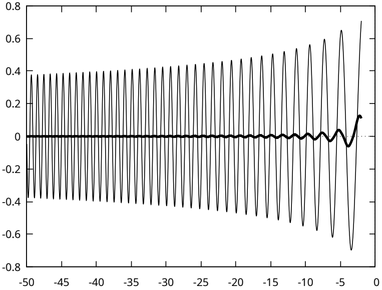


Figure: The numerical solution of the equation (10) on the figure practically coincides with the constructed asymptotic solution. Therefore, the figure shows the numerical solution, which is a thin line, and the difference between the numerical solution and the constructed asymptotic solution, which is a bold line near the abscissa axis.

The borderlines

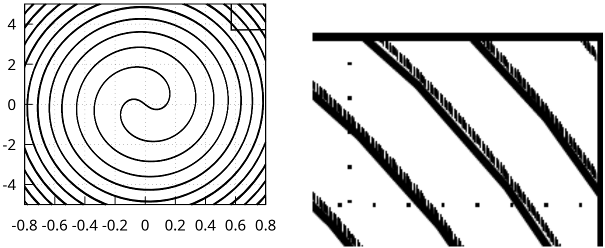


Figure: Here one can see the cross-section of the u, u' bifurcation boundary at $x = -50$ for solutions of the non-linearly perturbed Painlevé-2 equation (10) at $\varepsilon = 0.1$. The boundary obtained numerically for2048x4096 trajectories, with the beginning at $x = -50$, and the boundary calculated from perturbation theory almost coincide. Differences can be observed away from the center. The rectangle highlighted in the left drawing is enlarged in the right drawing. In the right drawing, the boundary obtained numerically corresponds to short vertical dashes, the boundary obtained by perturbation theory is indicated by continuous lines.

Conclusion

The equations for the parameters of the asymptotic behaviour of the Painlevé-2 transcendent at $x \to -\infty$ derived in 3 allow us to obtain a formula for the bifurcation boundary of solutions for a perturbed equation with a soft loss of stability in the neighbourhood of $x = 0$. This makes it possible to divide the solutions of the perturbed equation into solutions close to $\sqrt{x/2}$ and close to $-\sqrt{x/2}$ for $x \to \infty$. The results are illustrated by computing perturbations of various classes.

O.M. Kiselev An asymptotic structure of the bifurcation boundary of the perturbed Painleve-2 equation arXiv:2012.07895

Open problem