Lecture 9. Langevin and Fokker-Planck equations

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October 14, 2021

The Brownian motion and stochastic process

A Brownian motion is a model where main role plays stochastic collisions. A main idea of that motion is the stochastic process of the impact on a particle which we observe. This process assumes that there exist some continuous impacts with stochastic properties. To define the properties more clearly we need to use two new abstract concepts. One of them is a sigma-algebra and another one is a measure.

Countable sets

A set is called countable if the bijection between this set and the set of the natural numbers exists.
The simplest bijection can be constructed between Integer and natural numbers. Let's map all negative numbers at $0 > k \in \mathbb{Z} \to -2k + 1 \in \mathbb{N}$ and the positive numbers like $0 < k \in \mathbb{Z} \to 2k \in \mathbb{N}$, and $0 \in \mathbb{Z} \to 1 \in \mathbb{N}$.
Let us construct a bijection between the rational numbers $\mathbb{Q}$ and natural numbers $\mathbb{N}$. Consider the all rational fractions with the same sum of denominator and nominator. All these fractions numerate by a segment of natural numbers. For example fraction $1/1 \to 1$, then two fractions like $2/1 \to 3$, $1/2 \to 4$ and next fractions are following: $3/1 \to 5$, $2/2 \to 6$, $1/3 \to 7$. Such way gives us an opportunity to construct injection between the rational numbers and natural numbers.

Uncountable sets

An uncountable set is a set for which does not exist any map bijection between all elements of this set and the countable set.

Definition of σ-algebra

Let us consider a set Ω and some operator which take an element of this Ω.

- Consider the empty set $\emptyset$ and some subset X of a set Ω.
- If $A \subset \Omega$ then $\overline{A} \subset \Omega$.
- If $A_i \subset \Omega$ then $\cup_{i=1}^{\infty} A_i \subset \Omega$.

Such operation we consider as some elementary operation.
This elementary operation we will map a value which is a measure of this elementary issue with respect to measure of Ω.
One can consider a union of the elementary issues. Any such union of the elementary issues are subsets of the Ω.
Define all subsets of all unions of the subsets as an algebra Σ with respect to union as operator on the subsets as elements of set of all subsets.
Such algebra is called σ-algebra.
Other words we consider a subset and an operation of union of such subsets as an algebra.

Measure theory

Here we discuss an opportunity to construct a map $\Sigma \to \mathbb{R}$.

- For a discrete sets Ω with finite numbers of the elements the map is equal to a fraction of numbers issues in the subset $A \in \Sigma$ and numbers of all possible issues.
- For a discontinuous set we can consider the map as a fraction of the measure of the subset A with respect to all measure of the set Ω.

So we need to define a measure on the set Ω.

Measure theory

Consider an interval on the real axis $I \in \mathbb{R}$. The any subset A of this interval can be covered by a set of intervals with bounded length. The lowest bound of sum of lengths such covering intervals is called outer cover of the A.

The inner cover of the subset A is an outer cover of the subset $\overline{A}$ which is compliment set to this subset. An inner cover means the sum of the lengths which do not cover the outer cover of the compliment set $\overline{A}$.

If the lowest bound of outer cover and the higher bound of the inner cover coincide then the limit is called as measure of the subset A.

Cantor set

Let us consider the closed interval $[0,1]$. Divide the interval on three parts $[0,1/3]$, $(1/3,2/3)$ and $[2/3,1]$. Eliminate the middle open interval $(1/3,2/3)$. As a result we obtain the set:

$$C_1 = [0,1/3] \cup [2/3,1].$$

Now consider the intervals $[0,1/3]$ and $[2/3,1]$ divide both of them on three parts, and eliminate the middle paths, which are two open intervals $(1/9,2/9)$ and $(7/9,8/9)$. As a result we obtain the set

$$C_2 = [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1].$$

Repeat this procedure by dividing any closed interval on three parts of equivalent length and eliminating a central open interval.
The set defined by following formula

$$C = \cap_{k=1}^{\infty} C_k$$

is called the Cantor set.

The measure of the cantor set

let us consider the sum of lengths of the sets excluded from the Cantor set.

$$L = \frac{1}{3} + 2\frac{1}{9} + 4\frac{1}{27} + \cdots + 2^n\frac{1}{3^{n+1}} + \cdots =$$
$$\frac{1}{3}\left(1 + \frac{2}{3} + \frac{4}{9} + \dots\right) = \frac{1}{3}\frac{1}{1-\frac{2}{3}} = 1.$$

Hence the measure of the Cantor set is equal to 0.
The Cantor set is perfect set. That means for any neighbourhood of any point of this set contains an uncountable set of points of this set.

Langevin equation

Let us consider a particle in a position x on a real axis. The position of this particle changes with velocity

$$\dot{x} = -bx + \sigma\xi(t), \quad b, \sigma = \mathrm{const}\,.$$

Here parameter $\xi(x)$ we will call a stochastic process.
The solution of the Langevin equation:

$$x = x_0 e^{-bt} + e^{-bt} \int_0^t e^{b\tau} \xi d\tau.$$

Wiener process

The stochastic process modulates an influence a few impact on the particle by another very small particle or molecules with uniform distribution. Due to the central limit theorem we assume that the impacts of all these small particles or molecules give a normal distribution for their sum.

$$f(t, x) = \frac{e^{-x^2/t}}{\sqrt{2\pi t}}.$$

A Wiener process is the integral of the stochastic process

$$dw(x, t) = \xi(t), \quad w(0) = 0.$$

$$w(t) - w(s) = \frac{e^{-x^2/(2(t-s))}}{\sqrt{2\pi(x - s)}}.$$

Expectation and variation for the solution

- Mathematical expectation of x:

$$\mathrm{E}\, x = \mathrm{E}\, x_0 e^{-bt} + e^{-bt} \int_0^t e^{b\tau}\, \mathrm{E}\, \xi d\tau = x_0 e^{-bt} +$$

$$e^{-bt} \int_0^t e^{b\tau} \left(\int_{\mathbb{R}} x \frac{e^{-x^2/\tau}}{\sqrt{2\pi t}} dx \right) d\tau = x_0 e^{-bt}.$$

- Variance of x:

$$\mathrm{Var}\, x = e^{-bt} \int_{\mathbb{R}} \int_0^t e^{b\tau} x^2 \frac{e^{-x^2/\tau}}{\sqrt{2\pi\tau}} d\tau dx =$$

$$e^{-bt} \int_0^t e^{b\tau} \frac{2\tau}{\sqrt{\pi}} \int_{\mathbb{R}} \frac{x^2}{2\tau} e^{-x^2/(2\tau)} d\left(\frac{x}{\sqrt{2\tau}}\right) d\tau$$

$$= e^{-bt} \int_0^t e^{b\tau} 2\tau d\tau = \frac{t}{b} - \frac{1}{b} + \frac{e^{-bt}}{b^2}.$$

Therefore, for large value t we get:

$$\mathrm{Var}(x) \sim \frac{t}{b}, \quad \text{another form} \quad \langle x \rangle^2 \sim \frac{t}{b}.$$

A distribution of density

Definition

Define

$$\mathrm{Prob}(x(t) \in \mathbb{B}) = \int_{\mathbb{B}} u(t,z)dz.$$

$u(x,t)$ is a distribution of density for $x(t)$.

$$x(t+\Delta t) = x(t) + b(x(t))\Delta t + \sigma(x(t))[w(t+\Delta t) - w(t)].$$

Here $x(t)$ and $[w(t+\Delta t) - w(t)]$ we consider as two independent random values. Therefore to find the distribution of $x(t+\Delta t)$ we can consider the density

$$[w(t+\Delta t) - w(t)] = \frac{e^{-\frac{y^2}{2\Delta t}}}{\sqrt{2\pi\Delta t}}.$$

So to consider mathematical expectation we will integrate the distribution over new variable y.

Fokker-Planck equation

Let us consider a smooth and finite function $h(x)$. The mean value of this function can be obtained as:

$$Eh(x(t+\Delta t)) = \int_{\mathbb{R}} h(x)u(t+\Delta t, x)dx$$

On the other hand

$$h(x(t+\Delta t)) = h\left(x(t) + b(x(t))\Delta t + \sigma(x(t))\frac{e^{-\frac{y^2}{2\Delta t}}}{\sqrt{2\pi\Delta t}}\right)$$

and

$$u(x, t+\Delta t) = u(x,t)\frac{e^{-\frac{y^2}{2\Delta t}}}{\sqrt{2\pi\Delta t}}$$

Mathematical expectation for the solution of the Langevin equation

$$Eh(x(t+\Delta t)) \sim \int_{\mathbb{R}} dx \int_{\mathbb{R}} dy \left(h(x) + \frac{\partial h}{\partial x}(b\Delta t + \sigma y) + \frac{1}{2}\frac{\partial^2 h}{\partial x^2}(b\Delta t + \sigma y)^2 u(x,t) \frac{e^{-\frac{y^2}{2\Delta t}}}{\sqrt{2\pi\Delta t}} \right) \sim$$

$$\int_{\mathbb{R}} dx \left(h(x) + \frac{\partial h}{\partial x} b\Delta t + \frac{1}{2}\frac{\partial^2 h}{\partial x^2}\sigma \Delta t \right) u(x,t)$$

On the other hand we get:

$$Eh(x(t+\Delta t)) = \int_{\mathbb{R}} dx\, h(x)\, u(x, t+\Delta t)$$

$$\sim \int_{\mathbb{R}} dx\, h(x) \left(u(x,t) + \frac{\partial u}{\partial t}\Delta t \right).$$

Fokker-Planck equation

It yields:

$$\int_{\mathbb{R}} dx \left(\frac{\partial h}{\partial x} b \Delta t + \frac{1}{2} \frac{\partial^2 h}{\partial x^2} \sigma \Delta t \right) u(x,t) \sim \int_{\mathbb{R}} dx\, h(x) \left(\frac{\partial u}{\partial t} \Delta t \right).$$

Integrating by parts the left hand side and going to the limit $\Delta t \to 0$ one gets:

$$\int_{\mathbb{R}} dx h(x) \left(-\frac{\partial (bu)}{\partial x} + \frac{1}{2} \frac{\partial^2 (\sigma u)}{\partial x^2} \right) = \int_{\mathbb{R}} dx\, h(x) \left(\frac{\partial u}{\partial t} \right).$$

Then for the distribution $u(x,t)$ we get the Fokker-Planck equation:

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 (\sigma u)}{\partial x^2} - \frac{\partial (bu)}{\partial x}.$$

The bibliography

- W. Feller, Introduction to probability theory and its applications. Volume 1, Wiley, 1968.
- A. Lasota, M.C. Mackey, Probabilistic properties of deterministic systems. Cambridge University press, 1985.