Lecture 8. Random walk

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October 7, 2021

Random walk

Let us consider a walk on a set of integer numbers.

- Let x_0 be an initial point.
- The issue means a random choice $\xi \in \{-1, 1\}$.
- A step means a changing of position $x_{n+1} = x_n + \xi$.

We will consider following questions for this process.

- What is a mean value of a maximal distance between x_n and x_0 for N steps?
- How many times the process returns into initial stage, or the same, what is numbers of values k, such that $x_k = x_0$?
- At last, we will derive a diffusion equation as a limit for random walk with small steps and small time intervals.

A recurrent equation for the random walk

Let the probability of issue $\xi = 1$ be equal to p and a probability of opposite case is $q = 1 - p$.
Let us make n random steps. Then position $x = k$ will occur in two cases.

- First of them will occur if ξ be equal to 1 whereas $x = k - 1$. Such case have probability p.
- Second case will occur if ξ be equal to -1 whereas $x = k + 1$. Such case have probability q.
- Then a probability to be on position $x = k$ is:

$$u(n, k) = pu(n - 1, k - 1) + qu(n - 1, k + 1).$$

This equation is considered as a recurrent equation for the probability to occur in position $x =$ after n steps.

A mean value and a variance of the random walk

A mean value for one step is

$$E_1\xi = p - q.$$

Then for n steps we obtain the mean value as

$$E_n\xi = n(p - q).$$

Variance can be found as

$$Var = E(\xi - E\xi)^2 = E\xi^2 - (E\xi)^2 = 1 - (p - q)^2 = \\ (p + q)^2 - (p - q)^2 = 4pq.$$

Then the variance for n steps is

$$Var_n = 4pqn.$$

A return to the initial point

Let us consider n steps. k of that steps were made in positive direction. Then a current position is

$$x = k - (n - k) = 2k - n.$$

If this position is the origin,then $n = 2k$. The number of possibilities for k positive steps among $n = 2k$ steps is:

$$N = \frac{(2k)!}{(k!)^2}.$$

The probability for k steps in the positive direction and the same numbers steps in the negative direction is $p^k q^k$. Then we get:

$$P_k = Np^k q^k = \frac{(2k)!}{(k!)^2} p^k q^k.$$

The probability of return to origin at all steps is the sum over all trials:

$$P = \sum_{k=1}^{\infty} P_k = \sum_{k=1}^{\infty} \frac{(2k)!}{(k!)^2} p^k q^k.$$

if this sum exists!

Asymptotic behaviour of the number of returns

For a large value of k the factorial can be approximated with Stirling's formula:

$$k! \sim \sqrt{2\pi k}\frac{k^k}{e^k}.$$

This formula gives the asymptotic approximation of the probability of the returns after $2k$ steps:

$$P_k \sim \frac{\sqrt{4\pi k}(2k)^{2k}}{2\pi k k^{2k}} p^k q^k \sim \frac{2^{2k}}{\sqrt{\pi k}} p^k q^k = \frac{2^n}{\sqrt{\pi(n/2)}} p^{n/2} q^{n/2}.$$

The probability of return to origin at all steps is the sum over all trials:

$$P \sim \sum_{k=1}^{\infty} \frac{2^{2k}}{\sqrt{\pi k}} p^k q^k.$$

Asymptotic behaviour of the number of returns

Define $p = 1/2 - \delta/2$, $q = 1/2 + \delta/2$, $|\delta| < 1$, then:

$$P \sim \sum_{k=1}^{\infty} \frac{2^{2k}}{\sqrt{\pi k}} \left(\frac{1}{2} - \frac{\delta}{2}\right)^k \left(\frac{1}{2} + \frac{\delta}{2}\right)^k = \sum_{k=1}^{\infty} \frac{(1-\delta^2)^k}{\sqrt{\pi k}}.$$

Therefore if $0 < |\delta| < 1/2$, the probability for the return exists.
If $\delta = 0$ and hence $p = q = 1/2$ then the series diverges this means the random walk return to origin infinite times.

Random walk on a plane

The random walk on a plane means we should consider two random quantities. One of them defines the walk on the horizontal direction while the other one defines the walk on vertical direction. For simplicity we consider the walk with uniform probability for the steps in four different directions like up, down, left and right. Consequently the probability for return is equal to the following formula:

$$P_{k,l} = \frac{(2k)!}{(k!)^2}\frac{(2l)!}{(l!)^2}\frac{1}{4^n}, \quad 2k + 2l = n.$$

Using the Stirling's formula for large k, l we get:

$$P_{k,l} \sim \frac{1}{\pi\sqrt{kl}} = O(n^{-1}), \quad k = O(n), \quad l = O(n).$$

It yields for sum of probability of return as $n \to \infty$

$$P = \sum_{n}^{\infty} P_{k,l} = \infty.$$

This means the random walk on the plane returns to origin infinite times.

Random walk in three dimensional space

For three-dimensional space the same formula look in the another form:

$$P_{k,l} = \frac{(2k)!}{(k!)^2}\frac{(2l)!}{(l!)^2}\frac{(2m)!}{(m!)^2}\frac{1}{8^n}, \quad 2k+2l+2m=n.$$

Using the Stirling's formula for large k, l we get:

$$P_{k,l,m} \sim \frac{1}{\pi^{3/2}\sqrt{klm}} = O(n^{-3/2}), \quad k=O(n), \quad l=O(n), \quad m=O(n).$$

It yields for sum of probability of return as $n \to \infty$

$$P = \sum_n^{\infty} P_{k,l,m} \leq \mathrm{const}\,.$$

The value of this constant is ~ 0.53 it was obtained by Polya at 1921.

This means the random 3D walk does not obligatory return to the origin.

Random walk for different dimensions

The cause of difference of 3-dimensional case with respect to low dimensional cases is growth of number of possible directions for one steps: 2 possibilities for 1-dimensional, 4 possibilities for 2-dimensional and 8 possibilities for 3-dimensional walk.

Random walk and diffusion equation

Let us consider the step on the axis be Δx and the step on the time be Δt. Then the number of steps $n = t/\delta t$ in this case the mean value of the position

$$Ex \equiv Bt = (p - q)\Delta x(t/\Delta t).$$

The value of the variance is

$$Var(x) \equiv Dt = 4pq(\Delta x)^2(t/\Delta t).$$

Then the recurrent formula looks like follows:

$$u(x + \Delta x, t + \Delta t) = pu(x - \Delta x, t) + qu(x + \Delta x, t).$$

Random walk and diffusion equation

In this case a left hand-side of recurrent formula for the $u(x, n)$:

$$u(x, t + \Delta t) = u(x, t) + \frac{\partial u}{\partial t}\Delta t + O((\Delta t)^2),$$

and the right-hand side of the the recurrent formula is:

$$pu(x - \Delta x, t) + qu(x + \Delta x, t) = (p + q)u(x, t) -$$
$$p\frac{\partial u}{\partial x}\Delta x + q\frac{\partial u}{\partial x}\Delta x + (p + q)\frac{\partial^2 u}{\partial x^2}\frac{(\Delta x)^2}{2!} + O((\Delta x)^3).$$

It yields:

$$\frac{\partial u}{\partial t}\Delta t = -(p - q)\Delta x\frac{\partial u}{\partial x} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(\Delta x)^2.$$

Random walk and diffusion equation

Divide with Δt both sides of the equation.

$$\frac{\partial u}{\partial t} = -(p-q)\frac{\Delta x}{\Delta t}\frac{\partial u}{\partial x} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\frac{(\Delta x)^2}{\Delta t}.$$

Here substitute:

$$(p-q)\frac{\Delta x}{\Delta t} = B, \quad \frac{(\Delta x)^2}{\Delta t} = \frac{D}{4pq}.$$

It yields:

$$\frac{\partial u}{\partial t} = -B\frac{\partial u}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}, \quad \sigma^2 = \frac{D}{4pq}.$$

Fundamental solution of the diffusion equation

A function

$$v(x,t) = \frac{e^{-\frac{x^2}{2\sigma^2 t}}}{\sqrt{2\pi t}}$$

is a solution of the diffusion equation for $B = 0$.

$$\frac{\partial v}{\partial t} = -\frac{1}{2}\frac{e^{-\frac{x^2}{2\sigma^2 t}}}{\sqrt{2\pi t^3}} - \frac{x^2}{2\sigma^2 t^2}\frac{e^{-\frac{x^2}{2\sigma^2 t}}}{\sqrt{2\pi t}}.$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{x^2\, e^{-\frac{x^2}{2t\,\sigma^2}}}{\sqrt{2}\,\sqrt{\pi}\, t^{\frac{5}{2}}\,\sigma^4} - \frac{e^{-\frac{x^2}{2t\,\sigma^2}}}{\sqrt{2}\,\sqrt{\pi}\, t^{\frac{3}{2}}\,\sigma^2}$$

It is easy to see that substitution of these formula into the diffusion equation give a quantity.

The bibliography

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- A. Lasota, M.C. Mackey, Probabilistic properties of deterministic systems. Cambridge University press, 1985.