Lecture 7. Chaotic behaviour of dynamical systems

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Period of linear oscillations

An equation for linear oscillator can be written as follows:

$$\ddot{u} + \omega^2 u = 0.$$

Formally to find the period of motion we should multiply this equation by $\dot{u}$. in this case we get:

$$\frac{d}{dt}\left(\frac{\dot{u}^2}{2} + \omega^2 \frac{u^2}{2}\right) = 0.$$

Then we integrate this equation over t and obtain a formula:

$$\frac{\dot{u}^2}{2} + \omega^2 \frac{u^2}{2} = E, \quad \frac{du}{dt} = \pm\sqrt{2E - \omega^2 u^2}$$

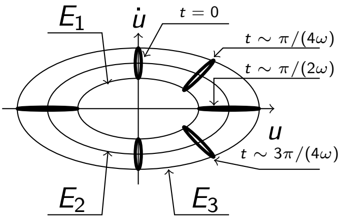
Here parameter E is a constant of integration in mathematical viewpoint and an energy in physical one.
This formula gives opportunity to integrate the equation:

$$t + t_0 = \int_{u_0}^{u} \frac{dy}{\pm\sqrt{2E - \omega^2 y^2}}$$

Period of linear oscillations

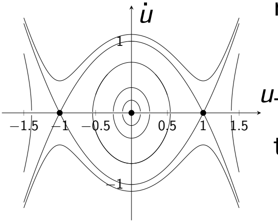
The period of oscillations can be obtained by the formula:

$$T = \frac{2}{\omega}\int_{-\frac{\sqrt{2E}}{\omega}}^{\frac{\sqrt{2E}}{\omega}} \frac{dy}{\sqrt{\frac{2E}{\omega^2} - y^2}} = \left| y = \frac{\sqrt{2E}}{\omega}x \right| = \frac{2}{\omega}\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \frac{2\pi}{\omega}.$$



So the period of linear oscillations does not depend on the energy.
If
we consider the motion with different initial values of the energy and $\dot{u} = 0$ on a phase plane we get the rotation with constant angle velocity ω.
A shape of initial cloud propagates and contracts during the circle.

Phase portrait for the solutions



One of the simplest equations for non-linear oscillation is a Duffing's oscillator:

$$\ddot{u} + 2u - 2u^3 = 0.$$

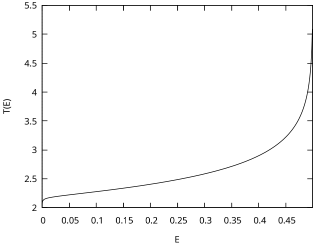
The same approach to integrating give us the follows formula:

$$\dot{u}^2 + 2u^2 - u^4 = 2E$$

This formula defines a closed curves as $0 < E < 1/2$ oscillating solution. If $E = 1/2$ one has eighth different solutions.

- There are two saddles at the following points$(-1, 0)$ and $(1, 0)$.
- Two separatrixies $u = \pm \tanh(t)$ separate infinite solutions and oscillations.
- Two unbounded separatrix moustaches $u = \pm \mathrm{cotanh}(t)$ which tend to $(-1, 0)$ and $(1, 0)$ as $t \to \infty$.
- Two unbounded separatrix moustaches $u = \pm \mathrm{cotanh}(t)$ which tend to $(-1, 0)$ and $(1, 0)$ as $t \to -\infty$.

Period of non-linear oscillations

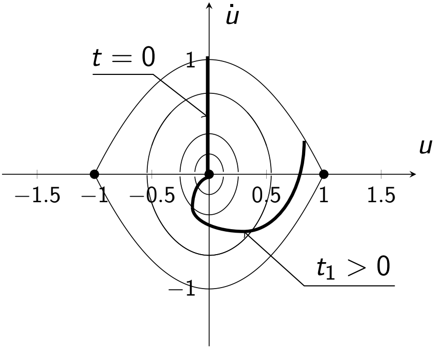


Let us consider
the oscillating solutions.
Period of the oscillations are

$$T = \int_{-u_1}^{u_1} \frac{dy}{\sqrt{2E - 2y^2 + y^4}},$$

where $u_1^2 = 1 - \sqrt{1 - 2E}$.
The dependency $T(E)$
one can see on the left picture.

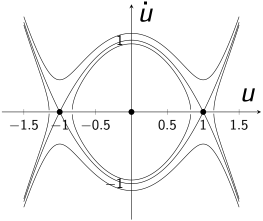
Lyapunov's instability of non-linear oscillations



The dependency of the period on the parameter E means that the points on the internal trajectories rotate faster that the points on the external trajectories. That is situated schematically on the left picture. On the picture the bold lines show the points at the same moments of time at $t = 0$ and at some $t_1 > 0$.

The points on the close trajectories disperse. This shows the Lyapunov's instability for the non-linear oscillations.

Solutions near separatrix



The separatries are special trajectories which tends from one saddle point to another.
A thin
layer on the phase plane contains trajectories with different behaviour. There are:

- the oscillations for $E < 1/2$,
- the separatricies for $E = 1/2$,
- the saddle points for $E = 1/2$,
- the unbounded trajectories for $E > 1/2$.

Therefore small variation of the parameter E changes the behaviour of solutions drastically.
We consider the Duffing's oscillator with small external force:

$$\ddot{u} + 2u - 2u^3 = \varepsilon \cos(\omega t + \Phi), \quad 0 < \varepsilon \ll 1.$$

and find the dependency of solution on this small perturbation.

Perturbation approach to solutions near separatrix

The upper separatrix solution looks like

$$u_0 = \tanh(t).$$

The solution near separatrix can be considered as a the separatix solution and additional small perturbation:

$$u \sim u_0 + \varepsilon u_1(t) + \sum_{n=2}^{\infty} \varepsilon^n u_n(t).$$

If one substitute this formula into the equation for Duffing's oscillator one get:

$$\ddot{u}_0 + 2u_0 - 2u_0^3 + \varepsilon(\ddot{u}_1 + 2u_1 - 6u_0^2 u_1) + O(\varepsilon^2) = \varepsilon \cos(\omega t + \Phi).$$

The $u_0 = \tanh(t)$ is the solution of unperturbed Duffing's oscillator. Therefore in this formula we have the terms of order ε and higher:

$$\varepsilon(\ddot{u}_1 + 2u_1 - 6u_0^2 u_1) + O(\varepsilon^2) = \varepsilon \cos(\omega t + \Phi).$$

Solution of the linearized equation

Let us consider for simplicity the terms of order ε^2. Then we consider the linearized equation:

$$\ddot{u}_1 + 2u_1 - 6u_0^2 u_1 = \cos(\omega t + \Phi).$$

Consider solutions for the complimentary linearized equation:

$$\ddot{v} + 2v - 6u_0^2 v = 0.$$

This equation can be obtained by derivation of the unperturbed Duffing's equation:

$$\frac{d}{dt}\left(\ddot{u}_0 + 2u_0 - 2u_0^3\right) = 0.$$

Linearly independent solutions of complementary equation

It yields:

$$\frac{d}{dt}\ddot{u}_0 + 2\dot{u}_0 - 6u_0^2\dot{u} = 0.$$

This means the $v_1 = \dot{u}_0 = 1/cosh^2(t)$ is a solution of the complimentary equation.
Linearly independent solution v_2 can be obtained using an equation for the Wronskian. For this linearized equation the Wronskian is a constant. Therefore we can consider the formula for the Wronskian as an equation for the v_2:

$$W(v_1, v_2) \equiv v_1\dot{v}_2 - \dot{v}_1 v_2 = 1.$$

Using the formula $v_1(t) = 1/\cosh^2(t)$ and the Wroskian one can obtain:

$$v_2 = \frac{\sinh(4t)}{32\cosh^2(t)} + \frac{\sinh(2t)}{4\cosh^2(t)} + \frac{3t}{8\cosh^2(t)}.$$

Behaviour of solution of linearized equation near separatrix

The solution of the the linear equation for n-th order of ε have a following form:

$$u_n = v_1(t) \int F_n v_2(\tau) d\tau - v_2(t) \int F_n v_1(\tau) d\tau.$$

For example $n = 1$ we have $F_1 \equiv \cos(\omega\tau + \Phi)$.
Formula for u_n is appropriated near upper separatrix. If we consider the solution which starts form small neighbour of the left saddle $(-1, 0)$ in the follow form:

$$u_n \sim A_n(k) v_1(t) + B_n(k) v_2(t), \quad t \to -\infty,$$

Then in near the right saddle point $(1, 0)$ we obtain

$$u_n \sim A_n(k+1) v_1(t) + B_n(k+1) v_2(t), \quad t \to +\infty,$$

The discrete map for the Duffing equation

$$\Delta B_1(k+1) = \int_{-\infty}^{\infty} \frac{\cos(\omega\tau + \Phi(k))}{\cosh^2(\tau)} d\tau = \frac{\pi \cos(\Phi(k))}{\cosh(\pi\omega/2)}.$$

$$B_1(k+1) = B_1(k) + \Delta B_1(k),$$

$$A_1(k+1) = \frac{1}{32}(B_1(k) + \Delta B_1(k)),$$

$$\Phi(k+1) = \frac{\omega}{2}\left(\log(\varepsilon) + \log\left(\frac{(-1)^{k+1}}{32}(B_1(k) + \Delta B_1(k))\right)\right) + \Phi(k),$$

$$B_n(k+1) = -32\frac{B_{n+1}(k) + \Delta B_{n+1}(k)}{B_1(k) + \Delta B_1(k)},$$

$$A_n(k+1) = \frac{B_1(k) + \Delta B_1(k)}{32}(A_{n-1}(k) + \Delta A_{n-1}(k)).$$

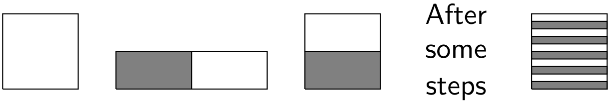
Instability of near the separatrix

This discrete map shows changing of the coefficients in the solution. The coefficients $B_n(k+1)$ are defined by $B_{n+1}(k)$. This means that the correction terms of the order ε^{n+1} define the correction terms of the lower order ε^n on the next step of the evolution.

Therefore the main term solution on the k-th step of evolution depends on the correction terms of the order ε^k on the first step. This means that the map is unstable.

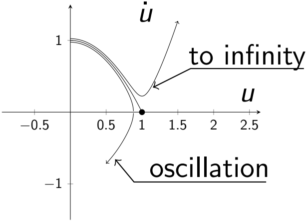
Opposite direction we see for the coefficients $A_n(k+1)$ which depends the coefficients $A_{n-1}(k)$ of the higher correction terms on the next step of the evolution.

The baker's map



So this the contraction in for the coefficients $A_n(k)$ and expansion for the coefficients $B_n(k)$ corresponds to the **baker's map** which is typical for the chaotic systems.

Horseshoe map



The trajectory oscillates near the separatrix while

$$-B_1(k)\cosh(\pi\omega/2) < \pi\cos(\Phi(k)).$$

The dependency of $\Phi(k)$ on the large value $\log(\varepsilon)$ shows additional instability with respect small parameter and behaviour of the solution.

Such behaviour corresponds to the **horseshoe map**.

The horseshoe map define the set like **Kantor set** for the oscillating trajectories with respect to values of ε on the small interval $\varepsilon \in (0, \varepsilon_0)$, where $\varepsilon > 0$.

Bibliography

Clear and detailed approach for perturbed Duffing equations near the separatrix can be found in:

S. G. Glebov, O. M. Kiselev, N. N. Tarkhanov, **Nonlinear Equations with Small Parameter. Volume 1 Oscillations and Resonances.** De Gruyter Series in Nonlinear Analysis and Applications.