Lecture 6. Laplace transform

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Functional equation for a Gamma-function

Let us construct a an integral formula for gamma-function. Denote an independent variable by z and an unknown function by $\Gamma(z)$. The equation for $G(z)$ looks like the formula for factorial:

$$\Gamma(z+1) = z\Gamma(z). \tag{1}$$

For natural argument $n \in \mathbb{N}$ the solution of (1) has a property $\Gamma(n+1) \equiv n!$.

Our aim is to find a solution of (1) for real value z. This equation is not solvable in terms of elementary functions. Below we will derive well-known integral formula for the gamma-function and study its asymptotic behaviour by Laplace-method.

The Laplace transform

We construct solution of(1) in the form of Laplace integral:

$$\Gamma(z) = \int_{-\infty}^{\infty} \tilde{\Gamma}(p) \exp(-pz) dp.$$

The right-hand side of (1) can be written in the form of:

$$z\Gamma(z) = z \int_{-\infty}^{\infty} \tilde{\Gamma}(p) \exp(-pz) dp =$$

$$-\tilde{\Gamma}(p) \exp(-pz) \Big|_{p=-\infty}^{p=\infty} + \int_{-\infty}^{\infty} \frac{d\tilde{\Gamma}(p)}{dp} \exp(-pz) dp. \qquad (2)$$

This formula is valid when limits for outside the integral term exist.

A differential equation for the image

Suppose

$$\lim_{p\to-\infty}\Gamma(p)\exp(-pz) - \lim_{p\to\infty}\Gamma(p)\exp(-pz) = 0.$$

We can check this property when a solution will be obtained. The left-hand side of (1) can be written in the form of:

$$\Gamma(z+1) = \int_{-\infty}^{\infty}\tilde{\Gamma}(p)\exp(-pz)\exp(-p)dp.$$

The equation (1) for direct image of gamma-function is:

$$\tilde{\Gamma}(p)\exp(-p) = \frac{d\tilde{\Gamma}(p)}{dp}.$$

A formula for the image

This equation can be solved easily by means of separating of variables:

$$\frac{d\tilde{\Gamma}}{\tilde{\Gamma}} = \exp(-p)dp,$$

or

$$\tilde{\Gamma} = -C\exp(-\exp(-p)), \quad \text{where parameter} \quad C \text{ does not depend on } p.$$

Let us check a property of outside the integral terms in (2), when $z > 0$.

$$\lim_{p\to-\infty} \exp(-\exp(-p))\exp(-pz) = 0;$$

$$\lim_{p\to+\infty} \exp(-\exp(-p))\exp(-pz) = 0.$$

It is easy to obtain an expression for an inverse image when the Laplace direct image was constructed

$$\Gamma(z) = C\int_{-\infty}^{\infty} \exp(-\exp(-p))\exp(-pz)dp.$$

Integral formula for the Gamma-function

This formula can be rewritten in short form. Change the variable of integration $t = \exp(-p)$. It leads to

$$\Gamma(z) = C \int_0^\infty \exp(-t) t^{(z-1)} dt.$$

- A special solution such as $C = \mathrm{const}$ and

$$\Gamma(1) = C \int_0^\infty \exp(-t) dt = 1.$$

  is called gamma-function.

Now we can write well-known integral representation for gamma-function:

$$\Gamma(z) = \int_0^\infty \exp(-t) t^{(z-1)} dt. \qquad (3)$$

Moivre-Laplace approximation for the gamma-function

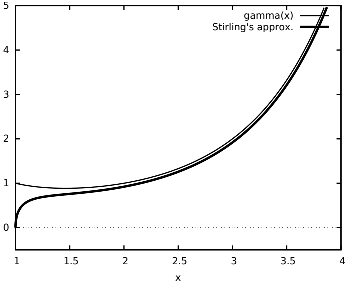

Figure: Gamma-function and Stirling-Moivre approximation.

To check that function (3) satisfies to (1) one can evaluate the integral from right-hand side of (3) through integration by parts:

$$\Gamma(z) = \frac{t^z}{z}\exp(-t)\Big|_{t=0}^{t=\infty} + \frac{1}{z}\int_0^{\infty}\exp(-t)t^z\,dt = \frac{1}{z}\Gamma(z+1).$$

The integral representation for gamma-function allows to evaluate an asymptotic expansion for gamma-function as $z \to \infty$.

$$\Gamma(z+1) \sim \sqrt{2\pi z}\left(\frac{z}{e}\right)^z.$$

Airy's equation

Airy's equation is called following ordinary differential equation of second order

$$u'' - zu = 0 \tag{4}$$

General solution of this equation oscillates as $z < 0$ and exponentially grows as $z > 0$. It is easy to explain such properties if one study an equation with fixed parameter instead of the varying parameter z. Solutions of equation

$$v'' - Zv = 0, \quad Z = \mathrm{const}, \tag{5}$$

are essentially dependent from a sign of parameter Z. The solutions oscillate with a frequency $\sqrt{-Z}$ when $Z < 0$. Otherwise when $Z > 0$ there are two linear independent solutions, one of them grows exponentially and another one decays exponentially. It is reasonable to expect the same behaviour for a solution of Airy equation on the intervals $z < 0$ and $z > 0$.

Graph for the Airy function

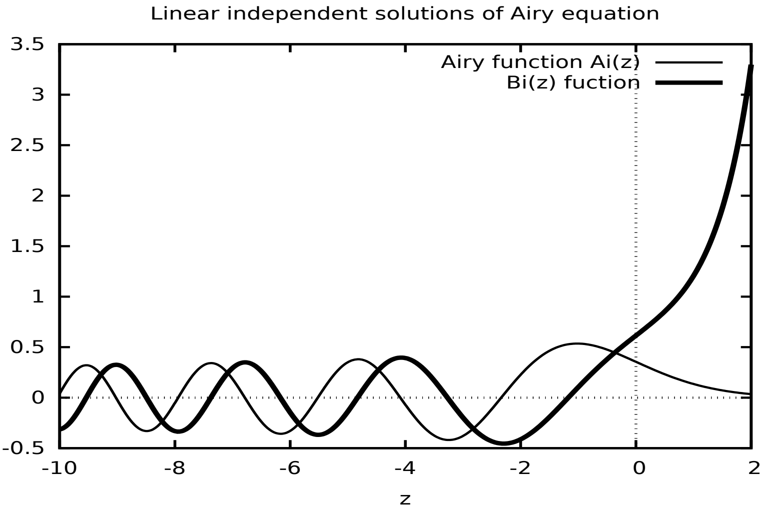

Figure: Two special solutions for the Airy equation.

An integral representation

A solution of Airy equation can be represented in an integral of Laplace type. Let us consider:

$$u(z) = \int_\gamma \tilde{u}(p) \exp(pz) dp.$$

Here γ is an unknown contour of integration. We choose this contour according to two conditions. The first condition is that the integral should be convergent. The second one is that the integral does not equal to zero. The contour will be determined after obtaining an explicit form for $\tilde{u}(z)$. Substitute the formula for $u(z)$ into Airy's equation. We suppose that a differentiation is possible under the integral sign:

$$\int_\gamma p^2 \tilde{u} \exp(pz) dp - z \int_\gamma \tilde{u} \exp(pz) dp = 0.$$

Equation for the image

It is convenient to remove a factor z via integration by parts:

$$z\int_\gamma \tilde{u}\exp(pz)dp = \tilde{u}\exp(pz)|_{\gamma_-}^{\gamma_+} - \int_\gamma \tilde{u}'\exp(pz)dp.$$

Points $\gamma_\pm$ are initial and end one of integration contour. We suppose that a sum of terms outside the integral equals to zero. It is possible when the contour is close or function $\tilde{u}(p)\exp(pz)$ equals to zero at these points. It yields:

$$\int_\gamma \left(p^2\tilde{u} + \tilde{u}'\right)\exp(pz)dp = 0.$$

The image of the Airy equation

The integral equals to zero when an expression in brackets equals to zero. It follows that,

$$p^2\tilde{u} + \tilde{u}' = 0.$$

This equation can be easy integrated:

$$\frac{d\tilde{u}}{\tilde{u}} = -p^2 dp.$$

or

$$\tilde{u} = C\exp(-p^3/3).$$

It leads to an integral expression:

$$u(z) = C\int_\gamma \exp(pz - p^3/3)dp.$$

The path of integration

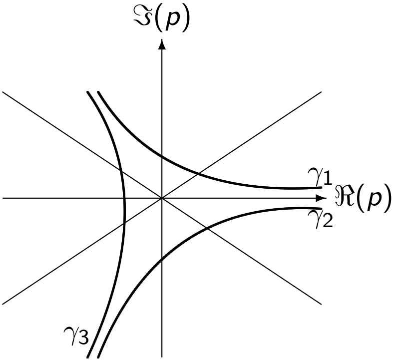

Figure: Curves $\gamma_1, \gamma_2, \gamma_3$ are contours of integration that lead to nontrivial Laplace integrals.

The integral is convergent when $\Re(p^3) < 0$ as $p \to \infty$. It means the contour should go to infinity inside sectors $-\pi/6 < \arg(p) < \pi/6$, $\pi/2 < \arg(p) < 5\pi/6$, $7\pi/6 < \arg(p) < 10\pi/6$. A contour of integration can be shrank to a point when both branches go to infinity through one sector. Such integrals equal to zero. To represent a non-trivial solution we choose contours that are going to infinity through two different sectors, see figure 3.

Integral formula for the Airy function

Consider the integral over contour γ_3. This contour is equivalent to an integral over image axis. Then

$$u(z) = C \int_{-i\infty}^{i\infty} \exp(pz - p^3/3)dp.$$

Change a variable of integration $p = ik$. It yields:

$$u(z) = iC \int_{-\infty}^{\infty} \exp(i(kz + k^3/3))dk.$$

The image part $\sin(kz + k^3/3)$ of integrand is an odd function with respect to k. The integral of it equals to zero. The real part $\cos(kz + k^3/3)$ of integrand is even with respect to k. It yields

$$u(z) = 2iC \int_{0}^{\infty} \cos(kz + k^3/3)dk.$$

A convergence of the integral

Show that the integral is convergent. Represent it as a sum of integrals:

$$u(z) = 2iC\int_0^{\sqrt{|z|}+1} \cos(kz + k^3/3)dk + 2iC\int_{\sqrt{|z|}+1}^{\infty} \cos(kz + k^3/3)dk.$$

The first integral is bounded for $z \in \mathbb{R}$. Consider the second one.

$$I(z) = 2iC\int_{\sqrt{|z|}+1}^{\infty} \cos(kz + k^3/3)dk.$$

Integrate it by parts:

$$I(z) = 2iC\int_{\sqrt{|z|}+1}^{\infty} \cos(kz + k^3/3)dk = 2iC\frac{\cos(kz + k^3/3)}{z + k^2}\Big|_{k=\sqrt{|z|}+1}^{k=\infty} +$$

$$+2iC\int_{\sqrt{|z|}+1}^{\infty} \frac{2k}{(z + k^2)^2}\sin(kz + k^3/3)dk.$$

The last integral is absolutely convergent. The term out of integral equals zero at $k = \infty$. It gives that the integral $I(z)$ is bounded and the integral $u(z)$ is bounded.

An integral formula for the Airy function

Let us choose $C = 1/2i\pi$. As a result we obtain a formula for Airy function:

$$Ai(z) = \frac{1}{\pi}\int_0^\infty \cos(kz + k^3/3)dk.$$

This function is a solution of Cauchy problem for Airy equation with an initial condition:

$$u(0) = \frac{1}{\pi}\int_0^\infty \cos(k^3/3)dk, \quad u'(0) = 0.$$

A value of Airy function at $z = 0$ is represented by gamma-function.

$$\frac{1}{\pi}\int_0^\infty \cos(k^3/3)dk = \frac{1}{\pi}\int_{-\infty}^\infty (\exp(ik^3/3))dk.$$

Denote $k^3/3 = t$. It gives $k^2dk = dt$ or $dk = dt/(3t)^{2/3}$, then

$$\frac{1}{2\pi}\int_{-\infty}^\infty \exp(ik^3/3)dk = \frac{1}{2\pi 3^{2/3}}\int_{-\infty}^\infty \frac{\exp(it)}{t^{2/3}}dt$$

An integral formula for the Airy function

Now the integral with respect to t can be represented as an integral over an imaginary axis: $\tau = it$, then

$$u(0) = \frac{1}{2\pi 3^{2/3}} \int_{-i\infty}^{i\infty} \frac{\exp(-\tau)}{(i\tau)^{2/3}} d(i\tau) = \frac{i}{\pi(3i)^{2/3}} \int_0^{i\infty} \frac{\exp(-\tau)}{\tau^{2/3}} d\tau$$

or

$$u(0) = \frac{-1}{\pi(3)^{2/3}} \int_0^{i\infty} \frac{\exp(-\tau)}{\tau^{2/3}} d\tau = \frac{-1}{\pi(3)^{2/3}} \Gamma(1/3).$$

An integral formula for the $Bi(z)$ function

Another linear independent solution with respect to $Ai(z)$ is obtained from an sum of integrals over the contours γ_1 and γ_2:

$$u(z) = C\int_{\gamma_1} \exp(pz - p^3/3)dp + C\int_{\gamma_2} \exp(pz - p^3/3)dp =$$

$$2C\int_0^{\infty} \exp(pz - p^3/3)dp +$$

$$C\int_0^{-i\infty} \exp(pz - p^3/3)dp + C\int_0^{i\infty} \exp(pz - p^3/3)dp.$$

Let us change a variable of integration in second and third integrals from the right-hand said of this formula $p = ik$. As a result we get:

$$u(z) = 2C\int_0^{\infty} \exp(pz - p^3/3)dp + 2C\int_0^{\infty} \sin(kz + k^3/3)dk.$$

The Airy's $Bi(z)$ integral is:

$$Bi(z) = \frac{1}{\pi}\int_0^{\infty} \Big(\exp(pz - p^3/3) + \sin(pz + p^3/3)\Big)dp.$$

Parabolic cylinder equation

A canonical form of parabolic cylinder equation is

$$y'' - \left(\frac{x^2}{4} + a\right)y = 0. \tag{6}$$

Here x is an independent variable, a is a parameter. This equation with a frozen large coefficient $x \to X = \mathrm{const}$ has two independent solutions. The first solution grows with respect to x and the second one decays for large value of x.

Another form for the parabolic cylinder equation

Parabolic cylinder equation can be written in another form:

$$y'' + \left(\frac{x^2}{4} - a\right)y = 0.$$

Solutions for this equation oscillate at large real values of x. Equation of the first form transfers to the second form through substitution:

$$x \to x \exp(i\pi/4), \quad a \to -ia.$$

Below we study solutions for parabolic cylinder equation written in the first form. Solutions of equation of the second form can be obtained by the given substitution.

Changing of the unknown function

To obtain an integral representation for the solution of parabolic cylinder equation it is convenient to change a required function :

$$y = \exp(-x^2/4)u.$$

Substitute this formula into equation. It gives an equation for u:

$$u'' - xu' - \left(\frac{1}{2} + a\right)u = 0$$

This equation is more convenient because it contains the first order of independent variable. It allows to obtain the first order equation for Laplace image and to integrate it.

The Laplace transform

We construct a solution u in the form of:

$$u(x) = \int_\gamma \exp(kx)\tilde{u}(k)dk.$$

This formula contains an unknown Laplace image $\tilde{u}(k)$ and unknown contour of integration in complex plane k.
We carry out formal calculations and suppose that they are valid.
We justify all manipulations when Laplace image $\tilde{u}$ and contour γ of integration are determined.

Evaluate formulae for the first and the second derivatives with respect to x:

$$\frac{d}{dx}\int_\gamma \exp(kx)\tilde{u}(k)dk = \int_\gamma k\exp(kx)\tilde{u}(k)dk.$$

$$\frac{d^2}{dx^2}\int_\gamma \exp(kx)\tilde{u}(k)dk = \int_\gamma k^2\exp(kx)\tilde{u}(k)dk.$$

Derivation of the integral formula

Substitute these formulae for equation and bring all terms under integral sign:

$$\int_\gamma \left(k^2 \tilde{u}(k) - xk\tilde{u}(k) - \left(\frac{1}{2} + a\right)\tilde{u}(k)\right) \exp(kx) dk = 0.$$

All terms besides $-xk\tilde{u}(k)$ in round brackets only depend on variable k . Transform the integral of this term by parts to remove a variable x:

$$-\int_\gamma (xk\tilde{u}(k)) \exp(kx) dk = -k\tilde{u}(k) \exp(kx)\Big|_{\gamma_-}^{\gamma_+} + \int_\gamma (\tilde{u}(k) + k\tilde{u}'(k)) \exp(kx) dk.$$

Where $\gamma_\pm$ are initial and finishing points for contour γ. We suppose that contour γ is such that a sum of out of integral terms equals to zero. It yields:

$$\int_\gamma \left(k^2 \tilde{u}(k) + \tilde{u}(k) + k\tilde{u}'(k) - \left(\frac{1}{2} + a\right)\tilde{u}(k)\right) \exp(kx) dk = 0.$$

Derivation of the integral formula

The integral equals zero when an integrand equals zero. This condition gives a differential equation:

$$k^2\tilde{u}(k) + k\tilde{u}'(k) + \left(\frac{1}{2} - a\right)\tilde{u}(k) = 0.$$

Solution of this equation is

$$\int \frac{d\tilde{u}}{\tilde{u}} = \int \frac{-k^2 + \left(a - \frac{1}{2}\right)}{k} dk$$

or

$$\tilde{u} = C\exp(-\frac{k^2}{2})k^{a-1/2}.$$

The solution can be represented in the form of:

$$y = C\exp(-x^2/4)\int_\gamma \exp\left(kx - \frac{k^2}{2}\right)k^{a-1/2}dk.$$

Derivation of the integral formula

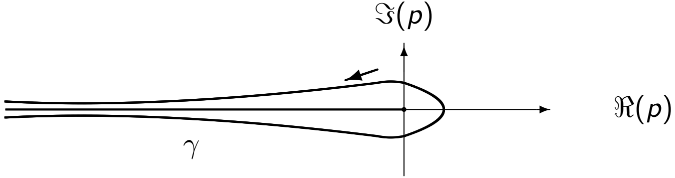

Figure: An integration over loop γ. The contour idents the point of branching $z = 0$ and goes under and over a crosscut $\Re(z) < 0$ that connects the point of branching $z = 0$ and infinity point.

The integrand has a singular point at $x = 0$ as $a < 1/2$. Let $k = \xi + i\lambda$, then $\Re(-k^2) = \lambda^2 - \xi^2$. It gives $\Re(-k^2) < 0$ as $\lambda^2 < \xi^2$. Any contour that contains point at infinity and goes through the sector $3\pi/4 < \arg(k) < 5\pi/4$ from infinity and goes back through $-\pi/4 < \arg(k) < \pi/4$ can be considered as γ. This contour does not include point $k = 0$. Cauchy theorem gives these contours are equivalent accurate to a loop around $k = 0$ that goes from infinity point under $\Im(k) = 0$ axis and goes back over $\Im(k) = 0$ axis.

The solution of the parabolic cylinder equation in special case

A point $x = 0$ is a pole of $n+1$-th order as $a = -n + 1/2, n \in \mathbb{N}$ and

$$y(x) = C\exp(-x^2/4)\int_\gamma \exp\left(kx - \frac{k^2}{2}\right)k^{-n}dk =$$

$$= C\exp(x^2/4)\int_\gamma \exp\left(-\frac{x^2}{2} + kx - \frac{k^2}{2}\right)k^{-n}dk =$$

$$= C\exp(x^2/4)\operatorname*{res}_{x=0}\left(\exp\left(-\frac{1}{2}\left(x-k\right)^2\right)k^{-n}\right) =$$

$$= C(-1)^n\frac{\exp(x^2/4)}{n!}\frac{d^n}{dx^n}\exp(-x^2/2).$$

A graph for the solutions.

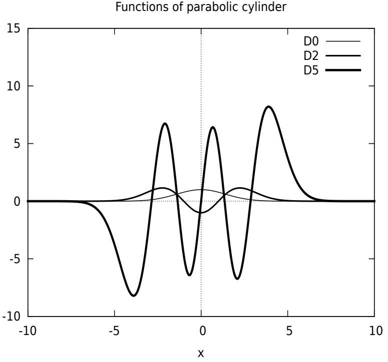

Figure: Graphs for parabolic cylinder functions. Values of parameter are $-a - 1/2 = 0$, $-a - 1/2 = 2$, $-a - 1/2 = 5$. A number of oscillations grows when parameter $-a - 1/2$ grows.

Summary

In this lecture we use the Laplace transform to obtain classical formulas for three type of functions.

- The gamma-function.
- The Airy function.
- The parabolic cylinder functions (Weber's functions).

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