Lecture 5. Elements of bifurcation theory

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Right-hand side with one additional parameter

Let us consider a differential equation

$$x' = f(c, x), \quad c \in \mathbb{R}.$$

We change the independent variable into new point $X = x - x_0(c)$, where

$$f(c, x_0(c)) = 0.$$

So in general case

$$\frac{\partial f}{\partial x} = f_1(c) \neq 0.$$

Hence in a small neighborhood of x_0 of the equation has the following form:

$$X' = f_1 X + O(X^2).$$

This means the point $X = 0$ is stable if $f_1 < 0$ and unstable if $f_1 > 0$.

The saddle-node bifurcation

For general dependency $f(c, X)$ might exist the point c_0 let say $c_0 = 0$ such that $f_1(c_0) = 0$. So one chooses new form of the equation:

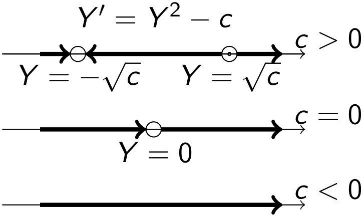
$$X' = f_2(c_0)X^2 + O(X^3).$$

This means the point $X = 0$ as $c = c_0$ became semi-stable. Namely, for $f_2 > 0$ the values $X(t) < 0$ tend to 0, where 0 is a equilibrium and $X(t) > 0$ goes away from the $X = 0$. Therefore $X = 0$ is a limit value for the trajectories $X(t) < 0$ and unstable point for the positive trajectories $X(t) > 0$. If $f_2 < 0$ we have a symmetric case.
In small neighborhood after scaling of the unknown function X we can write the equation locally in the form:

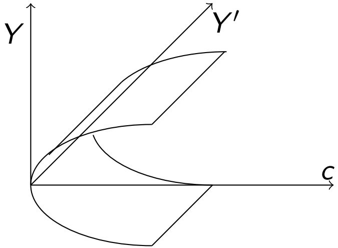
$$Y' = \pm Y^2 + c,$$

The saddle-node bifurcation

There are forth different cases. Consider $Y' = Y^2 - c$. If $c > 0$, then $Y = -\sqrt{c}$ and $-\sqrt{c}$ are equilibrium positions. The point $Y = -\sqrt{c}$ is stable equilibrium and another one $Y = \sqrt{c}$ is unstable.



The fold catastrophe



In the space of third parameters $Y' = Y^2 + c$ one get the folding. The opposite case $y' = -Y^2 + c$ has the same structure with contrast on sign.

The pitchfork bifurcation

Let us consider the equation with two independent parameters c_1 and c_2:

$$X' = f(c_1, c_2, X), \quad c_1, c_2 \in \mathbb{R}.$$

In general case there exists a pair of values c_1 and c_2 such that:

$$\frac{\partial f}{\partial x} \equiv f_1(c_1, c_2) = 0, \quad \frac{\partial^2 f}{\partial x^2} \equiv f_2(c_1, c_2) = 0.$$

In this case in small neighborhood of such point (X, c_1, c_1) the equation can be represented in their normal form:

$$Y" = (\pm Y^2 + c)Y, \quad c \in \mathbb{R}.$$

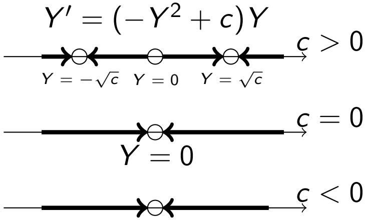
The chancing of the parameter c leads to changing of the equilibrium states.

The equilibrium states for pitchfork bifurcation

Let us consider one of
the cases of the pitchfork bifurcation:

$$Y' = (c - Y^2)Y.$$

In this case
due to the bifurcation arise two stable equilibrium states at $\pm\sqrt{c}$ and the equilibrium $Y = 0$ fall the stability.



The cusp catastrophe

The pitchfork bifurcation corresponds to the cusp catastrophe.
The normal form for the cusp catastrophe is:

$$z_1 = x_1^3 + x_1 x_2, \quad z_2 = x_1.$$

The Jacobian of the transformation looks like:

$$\begin{vmatrix} 3x_1^2 + x_2 & 0 \\ x_1 & 1 \end{vmatrix} = 3x_1^2 + x_2.$$

The line for the singular points is a cusp $x_1 = \pm\sqrt{-x_2/3}$.

The bifurcations of higher order

The next type of the bifurcation is the swallow tail:

$$\dot{x} = x^4 + a_2 x^2 + a_1 x + a_0, \quad a_i \in \mathbb{R}.$$

There is three parameter catastrophe:

$$z_1 = x_1^4 + x_2^2 x_2 + x_3, \quad z_2 = x_2, \quad z_3 = x_3.$$

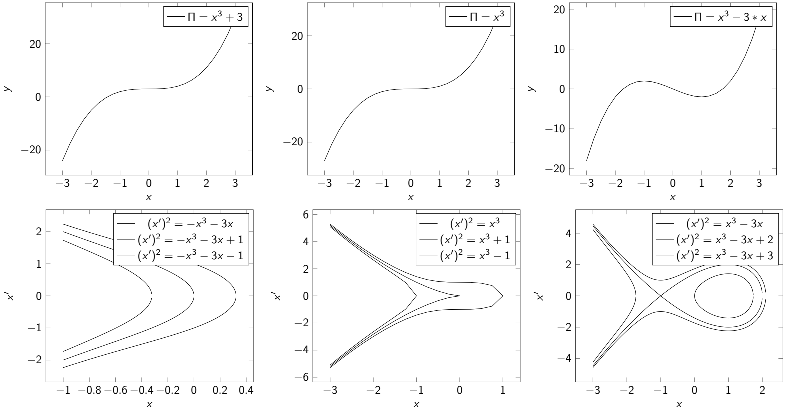
The Jacobian of the transformation looks like:

$$\begin{vmatrix} 4x_1^3 + 2x_2 x_1 + x_3 & 0 & 0 \\ x_2 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 4x_1^3 + 2x_2 x_1 + x_3.$$

The saddle-center bifurcation

Below one can see the origin of the saddle and center. in the equation

$$x'' = 3x^2 + c, \quad \text{or} \quad (x')^2 = x^3 + cx + c_2.$$



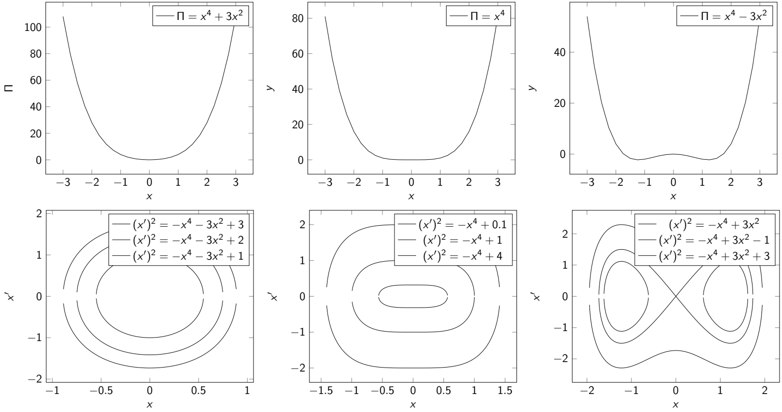
The Weierstrass elliptic function $\wp(x, g_2, g_3)$ is a special solution:

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3, \quad g_{2,3} \in \mathbb{R}.$$

The pitchfork bifurcation for second-order equation

Below one can see the pitchfork bifurcation in the equation:

$$x'' = 4x^3 + 2c_1 x, \quad \text{or} \quad (x')^2 = x^4 + c_1 x^2 + c_0.$$



The Jacobi function $\mathrm{sn}(x, k)$ is a special solution:

$$(\mathrm{sn}(x,k))^2 = (1 - \mathrm{sn}^2(x,k))(1 - k^2\,\mathrm{sn}^2(x,k)), \quad k \in [0,1).$$

Normal form for the van der Pol equation

The normal form of the van der Pol equation:

$$u'' + u = \nu(1 - u^2)u', \quad \nu \in \mathbb{R}.$$

If one use the normal form theory then one get the resonance equation:

$$q_1 - q_2 = 1.$$

Therefore the first term of the normal form as $q_1 = 2$, $q_2 = 1$ looks like:

$$\eta' = (i + \nu - \nu|\eta|^2)\eta.$$

This equation have the same behaviour as the Andronov-Hopf bifurcation:

$$z' = (i + \lambda + b|z|^2)z, \quad b = \alpha + i\beta, \quad \alpha, \beta, \lambda \in \mathbb{R}.$$

The Andronov-Hopf bifurcation: the origin of the limit cycle

Rewrite a solution of the equation

$$z' = (i + \lambda + b|z|^2)z, \quad b = \alpha + i\beta, \quad \alpha, \beta, \lambda \in \mathbb{R}.$$

in the exponential form:

$$z = re^{i\theta}, \quad r' + i\theta r = r(i + \lambda + (\alpha + i\beta)r^2).$$

It yields:

$$r' = \lambda r + \alpha r^3, \quad \theta' = 1 + \beta r^2.$$

- $\alpha < 0$, $\lambda < 0$ – stable origin;
- $\alpha < 0$, $\lambda > 0$, then $r^2 = -\lambda/\alpha$ stable limit cycle;
- $\alpha > 0$, $\lambda < 0$, then $r^2 = -\lambda/\alpha$ unstable limit cycle;
- $\alpha > 0$, $\lambda > 0$ – unstable origin.

Bibliography

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