Lecture 4. Normal forms and bifurcation theory

O.M. Kiselev

Innopolis University

September 2, 2021

A normal form in general case

The Henri Poincaré's idea to change the equation drastically constructing a map of the unknown function which allow to get the simplest form of the equation.
To find one we will consider a small neighbourhood of the given point for the equation and obtain coefficients of formal series which give transformation of the unknown function and allows rewrite the equation in the most simplest form.

History of the invention

Henri Poincaré gave the systematic approach to find the normal form of the function in his famous work "New methods of celestial mechanics" (1892).
He notices that such approach was used by Newcomb in his An investigation of the orbit of Uranus (1874) and Lindstedt in his famous work Bertrag zur integration Differentialgleichungen der Strórungsteorie (1883) published in Memories de l'Academie Imperiale des Sciences St.Petersburg.
Research on the theory of normal form for special cases, on the one hand, and the most general cases, on the other hand, continues to this day.

Changing of variables

The simplest and useful step is changing of the variables.

- A shift of unknown function.
- Scaling of independent variable like a time.
- Scaling of the unknown function.

Shift

Let us consider a map $\mathbb{R} \to \mathbb{R}$:

$$x' = f(x).$$

The aim for our resemble is to find the most simple form for this equation.
The special point of this equation is the point x_0 where $f(x_0) = 0$.
Then we change $X = x - x_0$ and it yields:

$$X' = f(X + x_0)$$

and $X = 0$ is a constant solution of the equation.

Time scaling

In small neighbourhood of origin:

$$X' = f'(x_0)X + O(X^2).$$

Next step is changing of the independent variable:

$$\tau = f'(x_0)t \quad \dot{X} = \frac{dX}{d\tau}.$$

It yields:

$$\dot{X} = F(X), \quad F = F(X)/f'(x_0), X^2 + O(X^3).$$

This equation in the small neighbourhood of the origin looks like:

$$\dot{X} = X + \frac{f''(x_0)}{2f'(x_0)}X^2 + O(X^3).$$

Scaling of unknown function

Next possibility to simplify the form of the equation is scaling the unknown function $X(\tau) = a\chi(\tau)$, where $a \in \mathbb{R}$ is a parameter of scaling:

$$a\dot{\chi} = a\chi + \frac{f''(x_0)}{2f'(x_0)} a^2\chi^2 + O(\chi^3).$$

Let the parameter a be reciprocal to the coefficient as X^2:

$$a = \frac{2f'(x_0)}{f''(x_0)}.$$

As a result we obtain a local form of the equation:

$$\dot{\chi} = \chi + \chi^2 + \frac{f'''(x_0)}{6} \left(\frac{f''(x_0)}{2f'(x_0)} \right)^2 \chi^3 + O(\chi^4).$$

Changing of variables

Resume. Changing of the variables allows to normalize some terms of the equation but it does not give a simplification of the equation.

A normal form in general case

Let us try to find the changing of the unknown function $X = \phi(y)$. It yields:

$$\phi' \dot{y} = F(\phi(y)).$$

Assume for simplicity that $F(X)$ is analytic and

$$F(X) = F_1 X + F_2 X^2 + F_3 X^3 + \dots.$$

The same form as a series of y for $\phi(y)$:

$$\phi(y) = y + \phi_2 y^2 + \phi_3 y^3 + \dots.$$

Then the equation for y can be written as a series:

$$\begin{gathered}(1 + 2\phi_2 y + 2\phi_3 y^2 + \dots)\dot{y} = \\ F_1(y + \phi_2 y^2 + \phi_3 y^3 + \dots) + \\ F_2(y + \phi_2 y^2 + \phi_3 y^3 + \dots)^2 + \\ F_3(y + \phi_2 y^2 + \phi_3 y^3 + \dots)^3 + \dots\end{gathered}$$

The coefficients for the $\phi(y)$

$$\dot{y} = \frac{y + (\phi_2 + F_1)y^2 + (F_3 + F_1\phi_3 + 2F_2\phi_2)y^3 + \dots}{1 + 2\phi_2 y + 2\phi_3 y^2 + 3\phi_4 y^3 + \dots}.$$

Now we use the formula for the sum:

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + z^4 - \dots.$$

It yields:

$$\dot{y} = F_1 y + (F_2 - F_1\phi_2)y^2 + (F_3 - 2\phi_3 + 2\phi_2^2)y^3 +$$
$$(F_4 + \phi_2 F_3 - 3\phi_4 + (7\phi_2 - 1)\phi_3 - 4\phi_2^2 + \phi_2^2)y^4 \dots.$$

So:

$$\phi_2 = \frac{F_2}{F_1}, \quad \phi_3 = \frac{F_1 F_3 + 2F_2}{2F_1^2}, \quad \dots.$$

Normal form

As a result we can claim:
For the first order ordinary differential equation with general analytic right-hand side one can obtain the local form:

$$\dot{y} = y + O(y^N), \quad \forall N \in \mathbb{N}.$$

Remark. We do not consider the problem of convergence for the series of $\phi(y)$.

Special case

Let us consider a special case of the right-hand side function:

$$\dot{X} = F(x), \quad F(x) = F_2 X^2 + F_3 X^3 + F_4 X^4 + \dots .$$

The the substitution $x = y + \phi_2 y^2 + \phi_3 y^3 + \dots$ yields:

$$\dot{y} = y^2 + F_3 y^3 + (F_4 + \phi_2 F_2 - \phi_3 + \phi_2^2) y^4 + \dots .$$

Then the normal form is follow:

$$\dot{y} = y^2 + F_3 y^3.$$

A diagonal form of the linear part

Let us consider the system of equations:

$$\dot{X} = AX + F(X), \quad AB_i = \lambda_i B_i, i = 1, 2$$

and $\lambda_1 \neq \lambda_2$, $\lambda_{1,2} \neq 0$. Define

$$X = BY, \quad B = [B_1, B_2].$$

$$\dot{Y} = B^{-1}ABY + B^{-1}F(BY).$$

$$\dot{Y} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} Y + B^{-1}F(BY).$$

Substitution

Let us construct the equations for Z, where $Y_i = Z_i + h_i(Z_1, Z_2)$ and

$$h_i(Z_1, Z_2) = \sum_{q_1+q_2>1} h_{i,q_1,q_2} Z_1^{q_1} Z_2^{q_2}, \quad q_i \in \mathbb{N}.$$

Let us redefine the components of the vector $B^{-1}F(B(Z + h(Z_1, Z_2))$ as a vector with components F_1 and F_2 :

$$B^{-1}F(B(Z + h(Z_1, Z_2)) = [F_1, F_2]^T.$$

These $F_{1,2}$ contain non-linear terms beginning of the second order with respect to vector Z.

Non-linear substitution

The derivation of Z on t looks like:

$$\dot{Y} = \dot{Z} + \frac{\partial h_i}{\partial Z_1}\dot{Z}_1 + \frac{\partial h_i}{\partial Z_2}\dot{Z}_2.$$

Let us substitute the representation for Y into the system of the equations. It yields:

$$\left(1 + \frac{\partial h_1}{\partial Z_1}\right)\dot{Z}_1 + \frac{\partial h_1}{\partial Z_1}\dot{Z}_2 = \lambda_1 Z_1 + \lambda h_1(Z_1, Z_2) + F_1(Z_1, Z_2),$$
$$\frac{\partial h_2}{\partial Z_1}\dot{Z}_1 + \left(1 + \frac{\partial h_2}{\partial Z_2}\right)\dot{Z}_2 + = \lambda_2 Z_2 + \lambda h_2(Z_1, Z_2) + F_2(Z_1, Z_2)$$

Matrix form of the equation for unknown functions

The system can be written in the matrix form:

$$\begin{pmatrix} \left(1+\frac{\partial h_1}{\partial Z_1}\right) & \frac{\partial h_1}{\partial Z_2} \\ \frac{\partial h_2}{\partial Z_1} & \left(1+\frac{\partial h_2}{\partial Z_2}\right) \end{pmatrix} \begin{pmatrix} \dot{Z}_1 \\ \dot{Z}_2 \end{pmatrix} =$$

$$\begin{pmatrix} \lambda_1 Z_1 + \lambda h_1(Z_1, Z_2) \\ \lambda_2 Z_2 + \lambda h_2(Z_1, Z_2) \end{pmatrix} + \begin{pmatrix} F_1(Z_1, Z_2) \\ F_2(Z_1, Z_2) \end{pmatrix}$$

The inverse matrix:

$$\begin{pmatrix} \left(1+\frac{\partial h_1}{\partial Z_1}\right) & \frac{\partial h_1}{\partial Z_2} \\ \frac{\partial h_2}{\partial Z_1} & \left(1+\frac{\partial h_2}{\partial Z_2}\right) \end{pmatrix}^{-1} = \frac{1}{D} \begin{pmatrix} \left(1+\frac{\partial h_2}{\partial Z_2}\right) & -\frac{\partial h_1}{\partial Z_2} \\ -\frac{\partial h_2}{\partial Z_1} & \left(1+\frac{\partial h_1}{\partial Z_1}\right) \end{pmatrix}$$

here

$$D = 1 + \frac{\partial h_1}{\partial Z_1} + \frac{\partial h_2}{\partial Z_2} + \frac{\partial h_1}{\partial Z_1}\frac{\partial h_2}{\partial Z_2} - \frac{\partial h_1}{\partial Z_2}\frac{\partial h_2}{\partial Z_1}$$

Let us multiply the system by this inverse matrix.

Equation for new function

$$\dot{Z}_1 = \frac{1}{D}\left(\lambda_1 Z_1 + \lambda_1 \frac{\partial h_2}{\partial Z_2} Z_1 - \lambda_2 \frac{\partial h_1}{\partial Z_2} Z_2 + \lambda_1 h_1 + \right.$$

$$\left. \lambda_1 h_1 \frac{\partial h_2}{\partial Z_2} - \lambda_2 h_2 \frac{\partial h_1}{\partial Z_2} + F_1 \frac{\partial h_2}{\partial Z_2} - F_2 \frac{\partial h_1}{\partial Z_2} + F_1 \right),$$

$$\dot{Z}_2 = \frac{1}{D}\left(\lambda_2 Z_2 + \lambda_2 \frac{\partial h_1}{\partial Z_1} Z_2 - \lambda_1 \frac{\partial h_2}{\partial Z_1} Z_1 + \lambda_2 h_2 + \right.$$

$$\left. \lambda_2 h_2 \frac{\partial h_1}{\partial Z_1} - \lambda_1 h_1 \frac{\partial h_2}{\partial Z_1} - F_1 \frac{\partial h_2}{\partial Z_1} - F_2 \frac{\partial h_1}{\partial Z_2 1} + F_2 \right),$$

and

$$\frac{1}{D} = 1 - \left(\frac{\partial h_1}{\partial Z_1} + \frac{\partial h_2}{\partial Z_2} + \frac{\partial h_1}{\partial Z_1}\frac{\partial h_2}{\partial Z_2} - \frac{\partial h_1}{\partial Z_2}\frac{\partial h_2}{\partial Z_1}\right) +$$

$$\left(\frac{\partial h_1}{\partial Z_1} + \frac{\partial h_2}{\partial Z_2} + \frac{\partial h_1}{\partial Z_1}\frac{\partial h_2}{\partial Z_2} - \frac{\partial h_1}{\partial Z_2}\frac{\partial h_2}{\partial Z_1}\right)^2 - \ldots .$$

Definition of the coefficients of formal series

We find the $h_i(Z_1, Z_2)$ a sum of the expansion by power of of the variables Z_1 and Z_2:

$$h_i(Z_1, Z_2) = \sum_{q_1, q_2 > 1} h_{iq_1q_2} Z_1^{q_1} Z_2^{q_2}.$$

Below we step by step will calculate the coefficients $h_{iq_1q_2}$.
If one substitute the formal series for h_i into the equation and equate the coefficients with the same terms $Z_1^{q_1} Z_2^{q_2}$ then one will see that the unknown coefficients $h_{1q_1q_2}$ and $h_{2q_1q_2}$ are defined one after another and the coefficients for this terms are obtained from the terms like $\lambda_i \frac{\partial h_j}{\partial Z_k} Z_n$. The other terms in the system of equation give addition terms with the quantities defines in previous steps.
To find the coefficient as unknown $h_{iq_1q_2}$ we use the series for the $1/D$ and obtain the coefficient for each $h_{iq_1q_2}$ in the obvious form.

Solvability condition

After the multiplication we obtain the follow terms:

$$-\lambda_1 q_1 h_{1,q_1 q_2} - \lambda_1 (q_2+1) h_{1q_1-1q_2+1} + \lambda_1 (q_2+1) h_{1q_1-1q_2+1} - \lambda_1 q_1 h_{1,q_1 q_2} + \lambda_1 h_{1,q_1 q_2} + H_{1,q_1 q_2} = 0.$$

Here the red terms appears due to the multiplication by the series defines $1/D$ and the additional term $H_{1,q_1 q_2}$ depends on known quantities.
Therefore we get:

$$(\lambda_1 q_1 + \lambda_2 q_2 - \lambda_1) h_{1,q_1 q_2} = H_{1,q_1 q_2}.$$

The same calculations for the second equation of the system give the same formula:

$$(\lambda_1 q_1 + \lambda_2 q_2 - \lambda_1) h_{2,q_1 q_2} = H_{2,q_1 q_2}.$$

Theorem about normal form (A.Poincaré)

Theorem

Two dimensional non-linear system:

$$\dot{x} = Ax + F(x),$$

can be rewritten in the form

$$\dot{Z}_i = \lambda_i Z_i + o(Z_1^n Z_2^m), \quad i = 1,2, \quad \forall n, m \in \mathbb{N}, \quad m + n > 1,$$

if the linear part of the system A has two non-zero eigenvalues $\lambda_1 \neq \lambda_2$ such that

$$\lambda_1 q_1 + \lambda_2 q_2 \neq \lambda_i, \quad i = 1,2$$

for any $q_1, q_2 \in \mathbb{N}$ and $q_1 + q_2 > 1$.

Theorem about normal form (A.Poincaré-Dulac)(1904)

Theorem

Two dimensional non-linear system:

$$\dot{x} = Ax + F(x),$$

can be rewritten in the form

$$\dot{Z}_i = \lambda_i Z_i + \sum_{q_1,q_2} H_{iq_1,q_2} Z_1^{q_1} Z_2^{q_2} + o(Z_1^n Z_2^m),$$

$$q_1, q_2 \in \mathbb{N},\ q_1 + q_2 > 1,\ i = 1,2, \quad \forall n, m \in \mathbb{N},\ m + n > 1,$$

where the linear part of the system A has two non-zero eigenvalues $\lambda_1 \neq \lambda_2$ *and*

$$\lambda_1 q_1 + \lambda_2 q_2 = \lambda_i, \quad i = 1,2$$

for all $q_1, q_2 \in \mathbb{N}$ *and* $q_1 + q_2 > 1$*.*

This theorem claims that the terms which can be eliminated one should use in the formula for the normal form of the equation.

Disadvantages

The method of normal form is effective for a lot of applications of differential equations. One of the most disadvantages for this approach is formal type of the series.
For the most of typical applications might be sufficient using a segment of the series.

Complex eigenvalues

Let us consider the system with complex conjugated eigenvalues. The system of the second order like follows:

$$u'' = -u + 2\alpha u' + f(u, u').$$

This equation in the system form looks like:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -2\alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ f(x_1, x_2) \end{pmatrix}$$

The eigenvalues of the matrix are:

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -2\alpha \end{vmatrix} = \lambda^2 + 2\alpha\lambda + 1 = 0, \quad \lambda_{1,2} = -\alpha \pm \sqrt{\alpha^2 - 1}.$$

Here we assume that $|\alpha| < 1$ and the eigenvalues are complex $\lambda_1 = \alpha + i\omega$, where $\omega = \sqrt{1 - \alpha^2}$.

Complex eigenvalues

The matrix with the eigenvectors looks like:

$$B = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}, \quad \begin{vmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{vmatrix} = \lambda_2 - \lambda_1 = -2i\omega.$$

and the inverse matrix are

$$B^{-1} = \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{pmatrix}.$$

Define new unknown vector Y as $X = BY$. The components of the vector Y are:

$$Y = B^{-1}X = \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} =$$

$$\frac{i}{2\omega} \begin{pmatrix} (-\alpha - i\omega)x_1 - x_2 \\ (\alpha - i\omega)x_1 + x2 \end{pmatrix} = \frac{1}{2\omega} \begin{pmatrix} (-i\alpha + \omega)x_1 - ix_2 \\ (i\alpha + \omega)x_1 + ix_2 \end{pmatrix}$$

This means the complex conjugation of y_1 and y_2: $y_1 = \overline{y_2}$

Complex eigenvalues

The system of equations for Y looks like:

$$\dot{Y} = B^{-1}ABY + B^{-1}[0, f]^T.$$

or in the shorter form:

$$\dot{Y} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} Y + \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} -f \\ f \end{pmatrix}.$$

Complex eigenvalues

New unknown variable is:

$$Y = Z + \begin{pmatrix} h_1(z_1, z_2) \\ h_2(z_1, z_2) \end{pmatrix}$$

Substitute this formula into the equation for Y. It yields:

$$\dot{z}_1 + \frac{\partial h_1}{\partial z_1}\dot{z}_1 + \frac{\partial h_1}{\partial z_2}\dot{z}_2 = \lambda_1(z_1 + h_1) - \frac{f}{\lambda_2 - \lambda_1}.$$

Assume $h_1 = \sum_{q_1 q_2} h_{1 q_1 q_2} z_1^{q_1} z_2^{q_2}$, $q_1, q_2 \in \mathbb{N}$, $q_1 + q_2 > 1$ and all coefficients are real.

After the some calculations like for the two-dimensional case and using the pure imaginary of the difference $\lambda_2 - \lambda_1 = -2i\omega$ we got the equation for $h_{1,q_1 q_2}$:

$$(-\omega q_1 + \omega q_2) h_{1,q_1 q_2} = \omega h_{1,q_1 q_2} + \frac{f_{1,q_1,q_2}}{2\omega},$$

where f_{1,q_1,q_2} are real and contains sums and multiplications of h_{1,p_1,p_2} where $p_1 + p_2 < q_1 + q_2$ and all h_{1,p_1,p_2} are known attitudes.

Complex eigenvalues

So, the resonant terms appears for the q_1 and q_2 such that:

$$q_2 - q_1 = 1, \quad q_{1,2} \in \mathbb{N}, \quad q_1 + q_2 > 1.$$

Theorem (Poincaré-Dulac for complex eigenvalues)

If the linear part of the real system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -2\alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ f(x_1, x_2) \end{pmatrix}$$

has complex conjugated eigenvalues $\alpha_{1,2} = \alpha \pm i\omega$, then the normal form looks like:

$$\dot{Z}_1 = \alpha Z_1 + \sum_{q_2 \in \mathbb{N}} h_{1,q_1,q_2} Z_1^{q_1} Z_2^{q_2}, \quad q_2 = q_1 + 1, \quad Z_2 = \overline{Z_1}.$$

Example

The Duffing equation:

$$\ddot{u} = -u + au^3, \quad a \in \mathbb{R}.$$

The linear part for this equation has pure imaginary eigenvalues $\pm i$ and the normal form looks like:

$$\dot{z}_1 = iz_1 + \sum_{q_1 \in \mathbb{N}} h_{q_1} |z_1|^{2q_1} z_1, \quad q_1 \geq 1.$$

Bibliography

Poincaré, H., New Methods of Celestial Mechanics (Am. Inst. of Physics, 1993).

Arnold, V.I., Geometrical Methods in the Theory of Ordinary Differential Equations (Springer-Verlag, New York, 1988).

Bruno, A.D., Local Methods in Nonlinear Differential equations (Springer-Verlag, Berlin, 1989)

Nayfeh, A.H., Method of Normal Forms. (Wiley, New York, 1993).