Lecture 3. Lyapunov's stability, limit cycles and discrete control

O.M. Kiselev

Innopolis University

August 26,2021

Stability

Let us consider a differential equation with the trivial solution

$$x' = f(x,t), \quad f(0,t) \equiv 0.$$

The trivial solution $x \equiv 0$ is stable if $\forall \varepsilon > 0$ and $\forall t_0$ $\exists \delta(\varepsilon, t_0) > 0$ and $\exists x_0$ $|x_0| < \delta \leq \varepsilon$ then

$$|x(t)| < \varepsilon, \quad \forall t > t_0.$$

Generally everyone must (sic!) understand that this defines stability as existence of general solution which will remain into any small neighbourhood of zero at all time after some t_0.

Example of stability and instability

- The trivial solution of the equation $x' = -x$ is stable. Let us take some $\varepsilon > 0$ and t_0 then for all $x_0 = \varepsilon$ and $\delta = \varepsilon$ the solution is $x(t) = \varepsilon e^{-(t-t_0)}$ and the condition of stability $|\varepsilon e^{-(t-t_0)}| \leq \varepsilon$ for all $t > t_0$.
- The trivial solution of the equation $x' = x$ is unstable. Let us take some $\varepsilon > 0$ and t_0 then for all $x_0 = \varepsilon$ and $\delta = \varepsilon$ the solution is $x(t) = \varepsilon e^{(t-t_0)}$ and the condition of stability is not valid: $|\varepsilon e^{(t-t_0)}| \geq \varepsilon$ as $t > t_0$.

First Lyapunov's theorem

Consider a system:

$$x' = Ax + \phi(x), \quad \phi(x) = o(x).$$

Theorem (First Lyapunov's theorem)

If all eigenvalues of of the matrix A have a negative real part, then $x \equiv 0$ are stable.

Rudely speaking this theorem uses the well-known property that for small values the linear part is more important than non-linear parts for any polynomials. Therefore if any solutions of the linearized system decrease, then the small general solution of the non-linear equation decreases also.

Example for the first Lyapunov's theorem

Let us consider a non-linear system of equations:

$$u' = v \quad v' = -u - bv + u^3.$$

The matrix A and the eigenvalues:

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -b \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \lambda \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

$$-\lambda(-b-\lambda) + 1 = 0, \quad \lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4}}{2}.$$

So the trivial solution is stable as $b > 0$.

The limitations of the using of the first Lyapunov's theorem

The first Lyapunov's theorem cannot be used without linear part like

$$u' = u^3.$$

and for the linear systems with non-negative real part of the eigenvalue like the follow:

$$u' = v, \quad v' = u, \quad \lambda = \pm\sqrt{-1}.$$

Second Lyapunov's theorem

Theorem (Second Lyapunov's theorem)

If $\exists L(x)$ *as* $|x| < \varepsilon$ *and*

- $L(x) = 0$ *if and only if* $x = 0$*;*
- $L(x) > 0$ *if and only if* $x \neq 0$*;*
- $\dot{L}(x) \leq 0$ *for all* $x < \epsilon$ *and* $t > t_0$*,*

Then the solution is stable.

The second Lyapunov's theorem assumes using some function L which can play role of distance between a general solution and the trivial one. And if such function does not grow this means that the solution remains in small neighbourhood of the trivial solution. Such function is called Laypunov function. The main difficulty is to find the Lyapunov function for certain system.
Often as the Lyapunov's function is convenient to use the energy or some of conservation law for the given system.

Examples for the first Lyapunov's theorem

- The Lyapunov system for the non-linear equation without any linear part:

$$u' = -u^3, \quad L(u) = u^2,$$
$$\frac{d}{dx}L(u) = 2u\frac{d}{dx}u = -2u^4.$$

Therefore $L(u) \equiv u^2$ is Lyapunov function for the given equation and the trivial solution is stable.

- The Lyapunov function for the linear system:

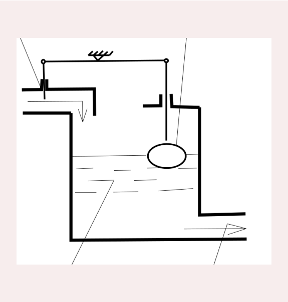
$$u' = v, \quad v' = -u, \quad L(u,v) = u^2 + v^2,$$
$$\frac{d}{dx}L(u) = 2u\frac{d}{dx}v + 2v\frac{d}{dx}u = 2uv - 2vu \equiv 0.$$

So $L(u,v) \equiv u^2 + v^2$ is Lyapunov function for the given system and the trivial solution is stable.

A float governor

Pozunov lived in Altay and he was the first man why construct two-cylinder machine in the world (1765). His steam machine needs a governor to control a level of water in the steam boiler. He used a float regulator.
The float regulator change the cross-section of the pipe which provide the water into the boiler.

Model of the float regulator.



Let the regular level of water equals to h.
The cross-section of the pipe changes as $k(h-x)$, where x is a current level of water.
Some part of water boil it out: mdt.
If $x<h$ then water balance defined by the equation:

$$dM = k_1(h-x)dt - mdt, \quad x<h.$$

Differential equation for the flow governor

Changing of water in the boiler:

$$dM = k_2 dx.$$

Then

$$k_2 dx = k_1(h - x)dt - mdt.$$

As result we obtain the differential equation for the governor:

$$\frac{dx}{dt} = X - kx, \quad k = \frac{k_1}{k_2}, \quad X = kh + \frac{m}{k_2}.$$

The constant solution:

$$x_s = h + \frac{m}{k_1}.$$

This solution is stable. Any other solution as $x < h$ grows:

$$x = x_s + x_0 e^k$$

History of invention

Van der Pol worked in Philips and constructed the generator for stable oscillations (1927). Now such kind of oscillation called relaxation oscillations. Such behaviour appears often in physics, technique and biology.

Generator

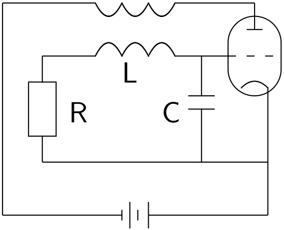


Figure: The electronic circuit for oscillator van der Pol's. This equipment generate a current with stable frequency. The cyclic recharging of the capacitor regulate the voltage on the triode. The anode current inducts the electromotive force in the secondary inductor. This current charges the capacitor as a result the voltage on the grid grows and the anode current decreases. The capacitor runs down. The anode current grows and electromotive force grows and such cycle repeated.

Notifications

M – coefficient of induction
I_a – anode current
E_g – the greed voltage
E_s – typical voltage
L – inductor
C – capacity
R – resistance
σ – conductivity of the triode.

Te anode current non-linear depends on the voltage on the grid:

$$I_a = \sigma \left(E_g - \frac{1}{3} \frac{E_g^3}{E_s^2} \right).$$

The voltage on the grid:

$$E_g = \frac{Q}{C}$$

Q is the charge of the capacitor.

The van der Poll's equation

The current is equal $I = (dQ)/(dt) = C(dE)/(dt)$. The equation for the voltage in the equipment

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = M\frac{dI_a}{dt}.$$

In dimensionless form: $U = E_g/E_s$, $t = T\tau$:

$$\ddot{U} + TR\sqrt{\frac{C}{L}}\dot{U} + T^2U = \frac{M\sigma T}{\sqrt{LC}}(1 + U^2)\dot{U}$$

Let A by the typical amplitude for the oscillations:

$$\ddot{y} - \nu(1 - y^2)\dot{y} + y = 0.$$

Instability near the origin

$\nu = M\sigma/\sqrt{LC} - R\sqrt{C/L} = M\sigma A^2/\sqrt{LC}, A = \sqrt{1 - \frac{RC}{M\sigma}}$
$\dot{y} = x$ $\ddot{y} = \dot{x}$, then

$$\frac{dx}{dy} = \frac{\nu(1-y^2)x - y}{x}.$$

The irregular point is equal $(0,0)$.

$$x^2 + y^2 = R^2$$

Then

$$R\frac{dR}{d\tau} = 2x^2\nu(1-y^2).$$

Therefore for small values of R the trajectories go out from the neighbourhood of the point $(0,0)$ and as $|y| > 1$ then R decrease.

Perturbation theory for small ν

Let us consider the case for small value of ν such as $0 < \nu \ll 1$. So let us construct the solution as a series of the parameter ν:

$$y \sim y_0(t) + \nu y_1(t) + \nu^2 y_2(t) + \nu^3 y_3(t) + \dots.$$

Substitute the formula for u into the van der Pol equation and collect the terms with the same order of the small parameter ν as a result we get:

$$y_0'' + y_0 + \nu(y_1'' + y_1 - (1 - y_0^2)y_0') + O(\nu^2) = 0.$$

The equation for y_0:

$$y_0'' + y_0 = 0$$

has a general solution in the form:

$$y_0 = a\cos(t - \phi),$$

where a and ϕ are parameters of the solution.

Perturbation theory for small ν

The equation for the correction y_1

$$y_1'' + y_1 = (1 - y_0^2)y_0$$

has more convenient form after the substitution of $y0 = a\cos(t-\phi)$:

$$y_1'' + y_1 = -(1 - a^2\cos^2(t-\phi))a\sin(t-\phi).$$

After using a trigonometric formulas we obtain:

$$y_1'' + y_1 = \left(\frac{a^2}{4} - 1\right)a\sin(t-\phi) + \frac{a^3}{4}\sin(3(t-\phi)).$$

The right-hand side of this equation contains the resonant term $\left(\frac{a^2}{4} - 1\right)a\sin(t-\phi)$, which leads to invalidity of the main term of the perturbation theory in the form $y_0 = a\cos(t-\phi)$ as $t = O(\nu^{-1})$.

Perturbation theory for small ν

To use the main term of the perturbation in the same form we must find a possibility to exclude the resonant term from the equation for y_1.
The method of exclusion of the resonant term is called van der Pol's method. We assume that the the parameter a is not a constant but it depends on new slow variable νt: $a = a(\nu t)$.
In this case:

$$\frac{d^2}{dt^2}y_0 = \frac{d}{dt}(\nu a' \cos(t-\phi) - a\sin(t-\phi)) =$$
$$\nu^2 a'' \cos(t-\phi) - \nu 2a' \sin(t-\phi) - a\cos(t-\phi).$$

Perturbation theory for small ν

This assumption leads to the new form of the equation for the first correction:

$$y_1'' + y_1 = \left(\frac{a^2}{4} - 1\right) a \sin(t-\phi) + \frac{a^3}{4}\sin(3(t-\phi)) + 2a' \sin(t-\phi).$$

This term allows to gather the coefficients of $\sin(t-\phi)$ into new equation for new additional function $a(\nu t)$:

$$a' = \left(1 - \frac{a^2}{4}\right) a.$$

For small value a the right-hand side is positive, then a grows. Therefore $a \equiv 0$ is unstable. If $a > 2$ then $a' < 0$ and then the function a decreases. So the value $a \equiv 2$ is the stable solution. Therefore

$$y \sim 2\cos(t-\phi)$$

looks as an attractor for all solutions as small values of ν. Such attractors are called *limit cycles*.

Construction

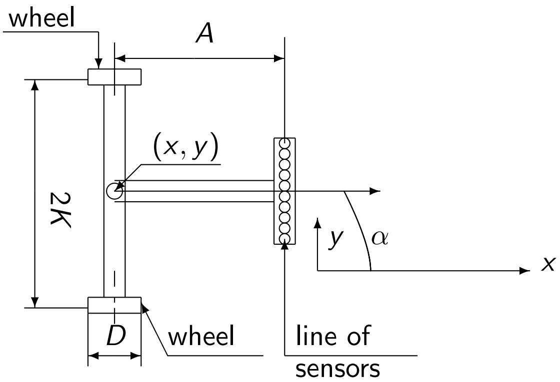


Figure: Coordinates of the center for the robot are (x, y) and the direction of the one is angle α. This coordinating triplet (x, y, α) defines the position the robot in the plane.

Kinematics

To derive the equations for the moving in the configuration space $(x, y, \alpha) \in \{\mathbb{R}^2 \times \mathbb{S}^1\}$ we define the angle velocity of left wheel by ω_L and the angle velocity of right wheel by ω_R. Typical values of the velocities are less than 100 rad/sec.

Let be known the angle velocities of the wheels and $\omega_L < \omega_R$.

Then the linear speed of left wheel is $\omega_L D/2$ and the linear speed of right wheel is $\omega_R D/2$.

Let us define a radius of the trajectory by R. Left and right wheels are moving along circumference. An equation for turn velocity of axis for the wheel is:

$$\omega_L D/2 = \dot{\alpha}(R - K), \quad \omega_R D/2 = \dot{\alpha}(R + K).$$

Then one can obtain the velocity of the turn and radius of the trajectory:

$$\dot{\alpha} = \frac{D}{4K}(\omega_R - \omega_L), \quad R = \frac{\omega_R + \omega_L}{\omega_R - \omega_L} K.$$

Kinematics

Let (X, Y) be coordinates of center for the turn. Current coordinates of the robot are (x_0, y_0, α_0). Then one can write the following formula for X, Y and R:

$$x_0 - X = R\sin(\alpha_0), \quad y_0 - Y = -R\cos(\alpha_0).$$

Let us define coordinates of the robot after the turn on the angle $d\alpha$ by (x_1, y_1, α_1). Then the change of the coordinates are:

$$\begin{aligned} \alpha_1 - \alpha_0 &= d\alpha, \\ x_1 - x_0 &= R(\sin(\alpha_1) - \sin(\alpha_0)), \\ y_1 - y_0 &= -R(\cos(\alpha_1) - \cos(\alpha_0)). \end{aligned}$$

Difference equations

Let us denote a time step by δ. Then a recurrent system of equations for coordinates can be written as follows:

$$\alpha_{n+1} - \alpha_n = \frac{D}{2K}\frac{\omega_R - \omega_L}{2}\delta,$$

$$x_{n+1} - x_n = \frac{\omega_R + \omega_L}{\omega_R - \omega_L} K(\sin(\alpha_{n+1}) - \sin(\alpha_n)),$$

$$y_{n+1} - y_n = -\frac{\omega_R + \omega_L}{\omega_R - \omega_L} K(\cos(\alpha_{n+1}) - \cos(\alpha_n)). \quad (1)$$

It is convenient to rewrite $x = KX$, $y = KY$ and to denote $\omega = (\omega_R + \omega_L)/2$, $\theta = (\omega_R - \omega_L)/2$, $\Delta = \delta\omega D/(2K)$ Then one obtain the system in dimensionless form:

$$\alpha_{n+1} - \alpha_n = \frac{\theta}{\omega}\Delta,$$

$$X_{n+1} - X_n = \frac{\omega}{\theta}(\sin(\alpha_{n+1}) - \sin(\alpha_n)),$$

$$Y_{n+1} - Y_n = -\frac{\omega}{\theta}(\cos(\alpha_{n+1}) - \cos(\alpha_n)).$$

Differential equation

One can rewrite the system of difference equations into system of ordinary differential equations where the independetn variable T is such that $T_{n+1} - T_n = \Delta$. Let us denote $\xi = \theta/\omega$, if $\Delta \to 0$ then one get:

$$\alpha' = \xi, \quad X' = \cos(\alpha), \quad Y' = \cos(\alpha). \tag{2}$$

Here $\xi = \theta/\omega$ is dimensionless control parameter. This parameter means the ratio the velocity of the turn and speed of moving along a trajectory of the robot.

Feedback control

If line-sensor shows that the deviation is equal Z, then the value of control parameter θ is :

$$\theta = \kappa_p Z.$$

Here $\kappa_p > 0$ is a coefficient of the feedback control.
A formula for the value of the deviation Z from the middle of the line-sensor can be obtained using this geometry property:

$$KZ\cos(\alpha) + KY + A\sin(\alpha) = 0. \tag{3}$$

Then the recurrent system for coordinates of the robot has a form:

$$\begin{aligned} \alpha_1 - \alpha_0 &= -\frac{\kappa_p}{\omega}\Delta\left(\frac{Y_0}{\cos(\alpha_0)} + \frac{A}{K}\frac{\sin(\alpha_0)}{\cos(\alpha_0)}\right), \\ X_1 - X_0 &= \frac{\omega}{\kappa_p}\frac{(\sin(\alpha_1) - \sin(\alpha_0))\cos(\alpha_0)}{\left(Y_0 + \frac{A}{K}\sin(\alpha_0)\right)}, \\ Y_1 - Y_0 &= -\frac{\omega}{\kappa_p}\frac{(\cos(\alpha_1) - \cos(\alpha_0))\cos(\alpha_0)}{\left(Y_0 + \frac{A}{K}\sin(\alpha_0)\right)}. \end{aligned} \tag{4}$$

Here one should consider ω/κ_p as a parameter of this system.

Differential equations for the controlled robot

The system of ordinary differential equations for this feedback control has a form:

$$\alpha' = -\frac{\kappa_p}{\omega}\left(\frac{Y}{\cos(\alpha)} + \frac{A}{K}\frac{\sin(\alpha)}{\cos(\alpha)}\right), \quad X' = \cos(\alpha), \quad Y' = \sin(\alpha). \quad (5)$$

One can rewrite this system as a linear differential equation of second order:

$$Y'' = -\frac{\kappa_p}{\omega}\left(Y + \frac{A}{K}Y'\right). \quad (6)$$

Let us change the independent variable $\tau = T\sqrt{\kappa/\omega}$ and let us use new parameter $\mu = (A/K)\sqrt{\kappa/\omega}$. As a result one obtain:

$$Y'' + \mu Y' + Y = 0.$$

A characteristic equation has a form:

$$\lambda^2 + \mu\lambda + 1 = 0.$$

Stability due to the first Lyapunov's theorem

Real parts of roots for this equation: $\Re(\lambda_{1,2}) < 0$. Therefore the zero is asymptotic stable solution for this equation. Then one get a following.

Theorem

The solution of system (5) $\alpha = 0$, $Y = 0$ and $X = \tau$ is asymptotic stable.

For fast linear speed or the same as $\mu \to 0$ the real part of the roots of characteristic equation is small and the stability property decreases. Really one can see:

$$k_1 \sim i - \frac{\mu}{2}, \quad k_2 \sim -i - \frac{\mu}{2}, \quad \mu \to 0.$$

Stability control

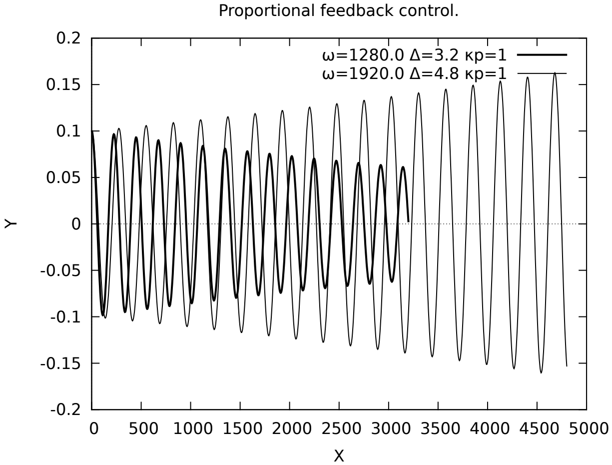


Figure: Stable and unstable straightforward moving under feedback control with different values of ω. The geometric dimensions of the robot are: $D = 0,05, K = 0,1,\ A = 0,2$. The step over time is $\delta = 0.01$. Coefficient of proportional control is: $\kappa_p = 1$.

Control with delay

The system with feedback control has a delay because of processing. Let us define a typical delay as δ.
The shift with respect of middle of line-sensor is know for time value $t-\delta$.
In terms of the variable T one get $\Delta = (D/2K)\delta\omega$.
The maps for α and Y are linked.

$$\alpha_{n+1} = \alpha_n + \frac{\kappa_p}{\omega} Z_n \Delta, \tag{7}$$

$$Y_{n+1} = Y_n - \frac{\omega}{\kappa_p} \frac{\left(\cos\left(\alpha + \frac{\kappa_p}{\omega} Z_n \Delta\right) - \cos(\alpha_n)\right)}{Z_n}, \tag{8}$$

where

$$Z_n = -\frac{Y + \frac{A}{K}\sin(\alpha_n)}{\cos(\alpha_n}.$$

Analysis of the discrete map

Let us obtain an asymptotic behaviour for this map near $(\alpha, y) = (0, 0)$:

$$\begin{pmatrix} \alpha_{n+1} \\ Y_{n+1} \end{pmatrix} \sim \begin{pmatrix} 1 - \frac{\kappa_p A}{\omega K}\Delta & -\frac{\kappa_p}{\omega}\Delta \\ \Delta - \frac{\kappa_p A}{\omega K}\frac{\Delta^2}{2} & 1 - \frac{\kappa_p}{\omega}\frac{\Delta^2}{2} \end{pmatrix} \begin{pmatrix} \alpha_{n+1} \\ Y_{n+1} \end{pmatrix}.$$

Let us consider quadratic form as a deviation:

$$F_n = \alpha_n^2 + \frac{\kappa_p}{\omega} Y_n^2.$$

One can obtain an asymptotic approximation when $\omega \to \infty$:

$$F_{n+1} \sim F_n - \frac{2A\kappa_p}{K\omega}\Delta\alpha_n^2 + \frac{\kappa_p}{\omega}\Delta^2\alpha_n^2$$

Therefore if

$$K\Delta - 2A < 0,$$

then $F_{n+1} < F_n$.

Condition for the control of fast robot

$$\omega < \frac{4A}{D\delta}.$$

The stability does not depend on proportional coefficient of the feedback control κ_p.

Summary

- The limit cycles show stable oscillations.
- The delay in the control add instability for the systems close to the border of its stability.