Lecture 2. Population dynamics

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Population dynamics

- Malthusian growth model
- Fibonacci growth model
- Logistic equation
- The predator-prey model

Mathematical models in population dynamics

The simplest and one of the first mathematical model of population was suggested by T.R. Malthus in his "An assay of the principle of population" published in 1798.

Let N be the numbers of the individuals in the population.The number of the new births is proportional of N. Then the growth of the numbers of individuals during Δt is equal to $\Delta N = kN\Delta t$. Here k is a proportional coefficient. So we obtain:

$$\frac{\Delta N}{\Delta t} = kN.$$

Let us consider $\Delta t \to 0$ then we obtain a differential equation:

$$\frac{dN}{dt} = kN.$$

The general solution of this equation is an exponent:

$$N = N_0 \exp(k(t - t_0)), \quad N|_{t=t_0} = N_0.$$

Such equation is one the simplest differential equations and therefore this equation is appeared in a lot of mathematical models.

Nuclear decay

The same law defines a nuclear decay and initial stage of epidemics. For the nuclear decay the proportional coefficient $k < 0$. As a rule physicists consider a half time of the decay T. It means the numbers of the atoms of the decayed quantity is:

$$N(t_0 + T) = N(t_0)/2, \quad \exp(kT) = 1/2, \quad k = -\log(2)/T.$$

One of the most dangerous product of the radioactive pollution is Iodine 131. The half period of the Iodine 131 equals $T_{I^{131}} = 8$ days. The half period of decay for uranium 232 is $T_{U^{232}} = 68,9$ years and the same period for uranium 238 $T_{U^{238}} = 4,468 \times 10^9$ years.

Fibonacci sequence

Well-know Fibonacci sequence appears as a solution of the population dynamics. Fibonacci considered a dynamics of the growth of rabbits (1202).
Every pair of the rabbits give an additional pair of rabbits due 2 mounts. Let us derive the formula for the quantity of the rabbit after $2n$ month. .
The sequence is defined by the rule:

$$N_{k+1} = N_k + N_{k-1}$$

Solution of the Fibonacci problem

Assume $N_k = \lambda^k$ then

$$\lambda^{k+1} - \lambda^k - \lambda^{k-1} = 0$$

$$\lambda^2 - \lambda - 1 = 0, \quad \lambda_{\pm} = \frac{1 \pm \sqrt{5}}{2}.$$

The linear combination of both solution is solution also:

$$N_k = c_1 \lambda_+^k + c_2 \lambda_-^k.$$

The problem is to find values of c_1 and c_2 using the initial data:

$$N_0 = c_1 + c_2 = 1$$

$$N_1 = c_1 \left(\frac{1 + \sqrt{5}}{2} \right) + c_2 \left(\frac{1 - \sqrt{5}}{2} \right) = 1$$

Solution of Fibonacci problem

$$c_1 + c_2 = 1, \quad \sqrt{5}(c_1 - c_2) = 1,$$

then

$$c_1 = \frac{1}{2} + \frac{1}{2\sqrt{5}}, \quad c_2 = \frac{1}{2} - \frac{1}{2\sqrt{5}}.$$

The final formula looks like:

$$N_k = \left(\frac{1}{2} + \frac{1}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^k + \left(\frac{1}{2} - \frac{1}{2\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^k.$$

After simplifications we obtain:

$$N_k = \frac{1}{2^{k+1}\sqrt{5}}\left((1+\sqrt{5})^{k+1} - (1-\sqrt{5})^{k+1}\right).$$

The logistic equation

The number of carps in the pond depends on connection between their food and the carps quantity. The limitation for the infinitely growth is a value of the foot in the pond.
Let us assume in the pond appears M quantity of the food, one carp eat m quantity of the food. If the food quantity is more than it needed for current numbers of the carps, then the numbers of the carps grow in the opposite case the carps extinct. Then the change of the carps in the pond is:

$$dN = k(M - mN)Ndt,$$

here k is proportional coefficient. So we obtain the differential equation:

$$\frac{dN}{dt} = k(M - mN)N.$$

Canonical form of the logistic equation

If the quantity $N > M/m$, then left hand side of the equation is negative and N decreases. If $N < M/m$, then the right-hand side is positive and N increases. The value $N_0 = M/m$ is the equilibrium of the carps numbers in the pond.
It is convenient to divide the left and right hand-side by N_0 and define new function as $n = N/n_0$. As a result we obtain:

$$\frac{1}{kM}\frac{dn}{dt} = (1-n)n.$$

Denote $\tau = kMt$ then we obtain a canonical form of the logistic equation:

$$\frac{dn}{d\tau} = (1-n)n.$$

This equation was studied by P.F. Verhulst at 1838.

Properties of the solution of $n' = (1-n)n$.

The trivial solution $n \equiv 0$ is unstable and all solutions in a small neighborhood are exponentially increase. To show this let us consider a linearized form of these equation for small n. After neglecting of the nonlinear term $-n^2$ as very small with respect to the linear term n we obtain:

$$n' \sim n. \quad n \sim n_0 \exp(\tau), \quad n_0 \in \mathbb{R}.$$

The trajectories go away from $n = 0$
To linearize the logistic equation near $n = 1$ we define:
$n = 1 + u(\tau)$. Here we assume the function $u(\tau)$ very small. Then the linear equation for $u(\tau)$:

$$u' = -u(1+u), \quad u' \sim -u, \quad u \sim u_0 \exp(-\tau), \quad u_0 \in \mathbb{R}.$$

The trajectories tends to $u = 0$ and hence $n = 1$ is stable solution.

General solution for the logistic equation

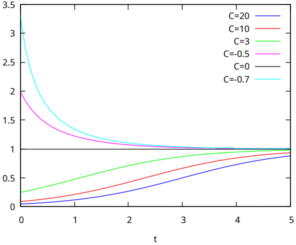

Figure: A general solution has the form:
$n(\tau) = \frac{1}{1+Ce^{-\tau}}, \quad \tau > 0, \quad C > -1.$

The formula for the solution of the logistic equation can be obtained by the following steps:

$$\frac{dn}{(1-n)n} = d\tau,$$
$$\log|n| - \log|1-n| = \tau - \tau_0,$$
$$\frac{n}{1-n} = \frac{e^\tau}{C}.$$

Here $|C| = e^{\tau_0}$
is a parameter of the solution.

The predator-prey model

A.J. Lotka (1925) and V. Volterra (1926) assumed the model with population of two kind like predators and preys.
Let x be numbers of preys and y be numbers of predators. The preys reproduced proportional their quantity and disappear proportional the numbers of the predators:

$$dx = (\alpha_1 x - \beta_1 yx)dt, \quad \alpha_1, \beta_1 > 0.$$

The numbers of the predators are increased proportional by the preys and disappeared proportional their quantity:

$$dy = (-\alpha_2 y + \beta_2 yx)dt, \quad \alpha_2, \beta_2 > 0.$$

As a result the system of the differential equations are:

$$\frac{dx}{dt} = (\alpha_1 - \beta_1 y)x,$$
$$\frac{dy}{dt} = -(\alpha_2 - \beta_2 x)y.$$

The simplest form of the predator-prey model

The points of equilibrium are $(x, y) = (0, 0)$ and $(x, y) = (\alpha_2/\beta_2, \alpha_1/\beta_1)$.
It is convenient to change the variables:

$$x = \frac{\alpha_2}{\beta_2} u, \quad y = \frac{\alpha_1}{\beta_1} v.$$

As a result we obtain:

$$\frac{du}{dt} = a_1(1 - v)x, \quad \frac{dv}{dt} = -a_2(1 - u)v.$$

The changing of the independent variable $t = \tau/a_1$ yields:

$$\frac{du}{d\tau} = (1 - v)u, \quad \frac{dv}{d\tau} = -k(1 - u)v.$$

Here $k = a_2/a_1$ is a parameter of the model.

The neighborhoods of equilibrium points of the predator-prey model

In the neighborhood of the origin the linearized system looks like:

$$\frac{du}{d\tau} \sim u, \quad \frac{dv}{d\tau} \sim -kv.$$

So the solutions are $u \sim u_0 \exp(\tau)$ and $v \sim v_0 \exp(-k\tau)$. The u exponentially grows therefore the trivial solution is unstable.
The linear equation in the neighborhood of the point $(u, v) = (1, 1)$ can be obtained after the changing of the variables:

$$u = X + 1, \quad v = Y + 1.$$

The linear system for X, Y has the form:

$$\frac{dX}{d\tau} = Y, \quad \frac{dY}{d\tau} = -kX.$$

To study the stability of the model in $(u, v) = (1, 1)$ we need additional calculations.

The conservation law for the predator-prey model

Let us divide the equation

$$\frac{dv}{d\tau} = -k(1-u)v,$$

by the equation

$$\frac{du}{d\tau} = (1-v)u.$$

As a result we obtain:

$$\frac{dv}{du} = \frac{-k(1-u)v}{(1-v)u}.$$

Then rewrite the equation in the differential form:

$$(1-v)\frac{dv}{v} = -k(1-u)\frac{du}{u}$$

or

$$\frac{dv}{v} - dv = kdu - k\frac{du}{u}.$$

After integrating we get:

$$\log(v) - v = -k\log(u) + ku + C.$$

The conservation law for the predator-prey model

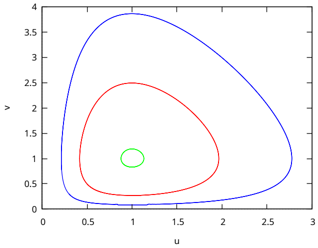

Figure: The phase portrait of the predator-prey model, $k = 2$.

The value

$$C = \log(vu^k) - (ku + v)$$

is a conservation law
for the predator-prey model:

$$\frac{dC}{d\tau} = \frac{dv}{d\tau}\frac{u^k}{vu^k} + k\frac{du}{d\tau}\frac{u^{k-1}v}{vu^k} - k\frac{du}{d\tau} - \frac{dv}{d\tau} = -k(1-u) + k(1-v) - k(1-v)u + k(1-u)v = -k + ku + k - kv - ku + kvu + kv - kuv = 0.$$