Lecture 1. Introduction into mathematical modelling

O.M. Kiselev

Innopolis University

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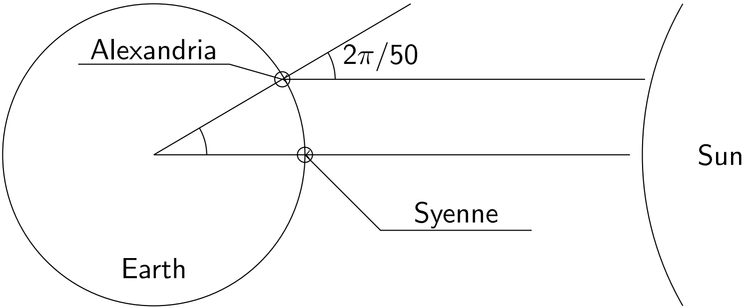
Mathematical modelling

- Mathematical modelling is not a science it is a methodology.
- Modelling is the origin of mathematics.

Eratosphenes of Cyrene (276 BC–195/94 BC)



The estimation of diam of the Earth (described by Cleomedes):

- Distance between Alexandria and Syene 5000 stadia
- The angle between the sun rays at Syene and Alexangria at the summer solstice equals $1/50$ of the arc.
- The diam of the Earth is equal 250000 stadia or about 39000 km.

Classic examples

- Isaac Newton shown that Kepler's laws are the sequences of the Newton's law of universal gravitation (1687). Newton invented differential equations on that way.
- Bernhard Riemann(1826 -1866) had studied the equations for a compressible gas and he predicted the existence of shock waves. The shock waves were found in an experiment later.
- J.C. Maxwell (1862-1864) had derived the equations for the electromagnetic fields and predicted the existence of the electro-magnetic waves. Later H. Herz (1887), who did not assume the Maxwell's theory, found such waves experimentally.
- A. Einstein shown, that the shift of the Perigee of the Mercury at 42 second per century can be explained by the relativistic theory of gravity.
- The Schroedinger equation is used to explain the electron spectrum of Hydrogen atom.

Principles of Mathematical modelling. Simplification

The model should include the most important traits.

- The mathematical pendulum on short periods of time:

$$\frac{d^2\phi}{dt^2} + \omega^2\phi = 0.$$

- The mathematical pendulum on large periods of time:

$$\frac{d^2\phi}{dt^2} + \omega^2 \sin(\phi) = 0.$$

- The mathematical pendulum on extra large periods of time:

$$\frac{d^2\phi}{dt^2} + \mu\,\mathrm{sgn}\left(\frac{d\phi}{dt}\right) + \omega^2 \sin(\phi) = 0.$$

The different levels of the simplification leads us to the hierarchy of the models for the same phenomenon.

Principles of Mathematical modelling. Dimensionless

- $gml\sin(\phi)$ – the torque of the gravity, where
  - ϕ – the angle in radian;
  - $g = 9,8\left[\frac{\mathrm{m}}{\mathrm{s}^2}\right]$ –acceleration;
  - $m = 3[\mathrm{kg}]$ – the mass of the pendulum;
  - $l = 0.5[\mathrm{m}]$ – the length of the pendulum;
- $ml^2\frac{d^2\phi}{dt^2}$, the moment of inertia
  - t – time in sec.

The sum of this torque yields:

$$3 \times 0.5^2\frac{d^2\phi}{dt^2}\left[\frac{\mathrm{kg\ m}^2}{\mathrm{sec}^2}\right] + 9.8 \times 3 \times 0.5\sin(\phi)\left[\frac{\mathrm{kg\ m}^2}{\mathrm{sec}^2}\right] = 0.$$

To obtain the dimensionless equation for the pendulum one should divide the equation on the typical values of the square of length and the mass.

$$\frac{d^2\phi}{dt^2} + \frac{9.8}{0.5}\sin(\phi) = 0$$

The terms of this equations have the dimension $\sec^{-2}$. The next step is the change of the time scale. For example if we are interested in the behaviour during a minute then $\tau = t/60$ and we obtain:

$$\frac{1}{60^2}\frac{d^2\phi}{d\tau^2} + \frac{9.8}{0.5}\sin(\phi) = 0, \quad \rightarrow \quad \frac{d^2\phi}{d\tau^2} + \frac{9.8 \times 60^2}{0.5}\sin(\phi) = 0.$$

The dimensionless coefficient in this equation is:

$$\omega^2 = \frac{9.8\frac{[\text{m}]}{[\text{s}^2]} \times 60^2[\text{s}^2]}{0.5[\text{s}]} \sim 7200$$

The same behaviour have all pendulums as $gT^2/l = 7200$.

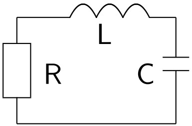
Principles of Mathematical modelling. Dimensionless

The dimensionless form of the equation give opportunity to use the same mathematical model for different physical phenomena.
For example the dimensionless form of the equation is used in the theory of the similarity. As the typical similarity models are considered the mechanical oscillating systems and electrical circuit.

Let U_k be the voltage

on the element $k \in \{C, R, L\}$.

$[\mathrm{B}] = \left[\frac{\mathrm{m}^2\ \mathrm{kg}}{\mathrm{A\ sec}^3}\right]$.



Let us consider the voltage

- on the capacitor: $U_C = \frac{Q}{C}$;
  - C – the capacitor dimension is : $[\mathrm{F}] = \left[\frac{\mathrm{A}^2\ \mathrm{sec}^4}{\mathrm{m}^2\mathrm{kg}}\right]$;
  - Q the charge dimension is colomb: $[\mathrm{C}] = [\mathrm{A\ sec}]$;

- The voltage on the resistance $U_R = RI$;
  - R the dimension of electrical resistance is : $[\mathrm{ohm}] = \left[\frac{\mathrm{m}^2\mathrm{kg}}{\mathrm{A}^2\mathrm{sec}^3}\right]$;
  - I the dimension of the current is ampere:[A];
- The voltage on the inductor: $U_L = L\frac{dI}{dt}$; L the dimension for the inductor is henry:
  - $[\mathrm{H}] = \left[\frac{\mathrm{m}^2\ \mathrm{kg}}{\mathrm{A}^2\ \mathrm{sec}^2}\right]$;

$$L\frac{dI}{dt}\left[\frac{\mathrm{m}^2\ \mathrm{kg}}{\mathrm{A}\ \mathrm{sec}^3}\right] + IR\left[\frac{\mathrm{m}^2\mathrm{kg}}{\mathrm{Asec}^3}\right] + \frac{Q}{C}\left[\frac{\mathrm{m}^2\mathrm{kg}}{\mathrm{A}\ \mathrm{sec}^3}\right] = 0.$$

Let us define $t = T\tau$, $I = (dQ)/(dt)$, $r = TR/L$, $Q = \overline{Q}q$.

$$\frac{d^2q}{d\tau^2} + r\frac{dq}{d\tau} + \frac{T^2}{LC}q = 0.$$

Principles of Mathematical modelling. Validity

- The mathematical model must be valid. This means that the model must compared with the experiment.
- The parameters of the model must belong to the defined intervals.

The example:

- Linear equation for the oscillations is valid for short interval of time and for small amplitudes.
- Non-linear equation for the oscillations is valid for short interval of time.
- The non-linear and dissipative equation for the oscillation is valid for long times.

Classifications of the mathematical modelling

To stay in the formal way we must mention several approach to classification of the mathematical models.

- There are linear and nonlinear models. The nonlinear and linear oscillators belong to different type of the models.
- There are deterministic and probabilistic models. The heat equation and the Boltzmann equation show these different approaches.
- There are deductive and phenomenological models. The kinetic theory of gases and the gas laws like Boyle–Mariotte law and Gay-Lussac's law show these opposite approaches in the modelling.

The validity of linear model

The equation for the pendulum:

$$\phi'' + \sin(\phi) = 0.$$

The Taylor series for the sine:

$$\sin(\phi) = \phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \dots .$$

Assume that the value of $\phi = \varepsilon u$, where $0 < \varepsilon \ll 1$. Then the equation for the pendulum can be written in the form:

$$u'' + u - \varepsilon^2 \frac{u^3}{3!} + \varepsilon^4 \frac{u^5}{5!} - \dots = 0.$$

Let us consider the interval of validity of the linear model for this equation:

$$u'' + u = 0.$$

The general solution for the linear model:

$$u = a \sin(t + \alpha), \quad a \in \mathbb{R},\ \alpha \in [0, 2\pi).$$

The validity of linear model

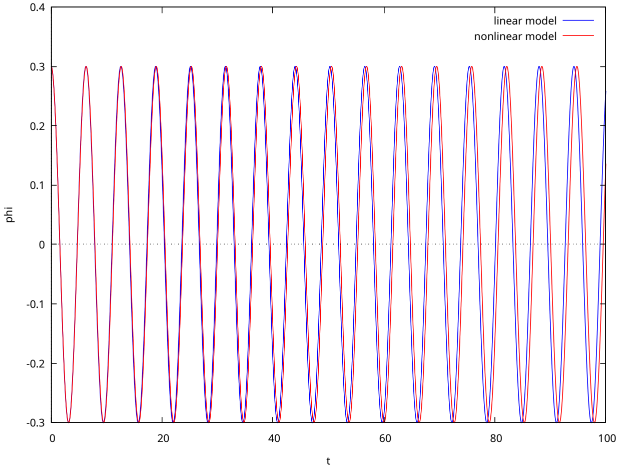


Figure: One can see the difference between linear and non-linear pendulum model at large time.

The validity of linear model

If we consider the solution as two terms of the perturbation theory:

$$\phi \sim \varepsilon u_1 + \varepsilon^3 u_2,$$

substitute this formula into the pendulum equation, then we get:

$$\varepsilon u_1'' + \varepsilon^3 u_2'' + \varepsilon u_1 + \varepsilon^3 u_2 - \varepsilon^3 \frac{u_1^3}{3!} + \cdots = 0.$$

Gather the terms with the same order of ε:

$$\varepsilon(u_1'' + u_1) + \varepsilon^3 \left(u_2'' + u_2 - \frac{u_1^3}{3!} \right) + \cdots = 0.$$

Here ε is independent parameter, so the equation is valid if the terms of all order of ε are equal to 0.

The validity of linear model

It yields two equations for u_1 and u_2:

$$u_1'' + u_1 = 0, \quad u_2'' + u_2 = \frac{u_1^3}{3!}.$$

First equation have a general solution:

$$u_1 = a_1 \sin(t + \alpha_1), \quad a_1 \in \mathbb{R},\ \alpha_1 \in [0, 2\pi).$$

Second equation can be written as

$$u_2'' + u_2 = \frac{a_1^3}{6} \sin^3(t + \alpha_1).$$

Using trigonometric formula we get:

$$sin(3\beta) = (1 - cos^2(\beta))sin(\beta) = sin(\beta) - cos^2(\beta)\sin(\beta) =$$
$$\sin(\beta) - \frac{1}{2}\cos(\beta)\sin(2\beta) = \sin(\beta) - \frac{1}{4}\sin(3\beta) - \frac{1}{4}\sin(\beta) =$$
$$\frac{3}{4}\sin(\beta) - \frac{1}{4}\sin(3\beta).$$

The validity of linear model

It yields the equation for u_2 in the form:

$$u_2'' + u_2 = \frac{a_1^3}{6}\frac{3}{4}\sin(t+\alpha_1) - \frac{a_1^3}{6}\frac{1}{4}\sin(3(t+\alpha_1)).$$

The general solution for this equation has a form:

$$u_2 = a_2\sin(t+\alpha_2) + \frac{a_1^3}{6}\frac{3}{8}t\cos(t+\alpha_1) + \frac{1}{8}\frac{a_1^3}{6}\frac{1}{4}\sin(3(t+\alpha_1)).$$

The red term grows linearly with respect to t. Therefore this term have the order ε as $t = O(\varepsilon^{-2})$.
Hence the linear approximation of the pendulum equation is not valid on the large value of t, namely as $t = O(\varepsilon^2)$.