Klein's model of Lobachevskii geometry and pseudosphere

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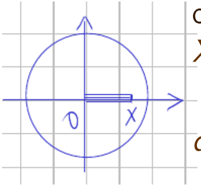
Axiom about existence of segment with given length

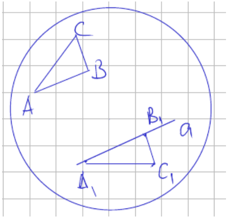


The length
of the interval between origin $O(0,0)$ and point $X(x,0)$. due to the formula for the distance:

$$d = \left| \log \left(\frac{-1-0}{-1-x} : \frac{1-0}{1-x} \right) \right| = -\log \left(\frac{1-x}{1+x} \right).$$

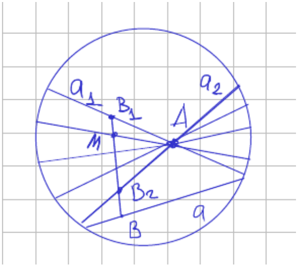
So, one can construct an interval for given distance.

Axiom of existence of given triangle



Let us consider the triangle ABC.
The **axiom** said
that there exist an equivalent triangle $A_1B_1C_1$ on a given straight line a and given points $\{A_1, B_1\} \in a$ and the point C_1 belongs to given side of the line a.
The same axiom has the Klein's model of the Lobachavskii geometry.

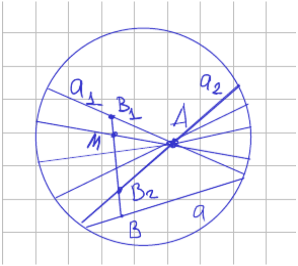
Axiom of parallel lines in Lobachavskii geometry



There exist a line *a* and a point $A \notin a$ such that two different lines pass through the point A and non-intersect with given line a.

Lemma. So uncountable numbers of lines pass through the point A and do not intersect with the line *a*.

Proof of existence diverge lines

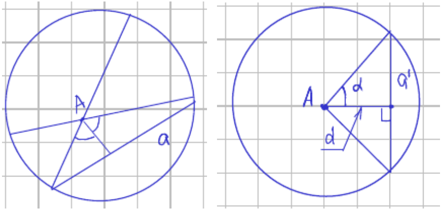


Let a_1 and a_2 be two lines and $a_{12} \cup a = \emptyset$. Consider points $B_1 \in a_1$ and in that side with respect to the point A and a point $B \in a$ such that the point $B_2 = (B, B_1) \cap a_2$ belongs to the interval $[B_1, B]$. For every point $M \in (B_1, B_2)$ the straight line $(A, M) \notin a$ because in an opposite case to intersect the straight line a the straight line (A, M) should cross the line a_2 in two different points in the point A and in one another point.

Parallel lines in the Klein's model

Set us define a straight line a' as a **parallel line** to given line a if such straight line is a border line in the set of lines which do not intersect to the line a.
Other lines which do not intersect with the line a are called **divergent lines**.

A Lobachevskii function



Let us consider the given straight line a and their parallel line a' which passes trough the point $A \notin a$. Define an acute angle λ between a parallel line a' and an orthogonal line passed trough the point A as an **angle of parallelism**.

Lemma. The angle of parallelism is uniquely defined.

Proof. The angular value can be obtained if one consider the vertex of the angle of parallelism as a central angle. In this case both angles of parallelism have the same angular value.

A Lobachevskii function

Lemma. The angle of parallelism depends on a distance between the point A and the straight line a.
A function defines the dependence an angle of parallelism of the distance between a point A and given straight line a is celled a Lobachevskii function:

$$\cos(\alpha) = x, \quad d = -\log\left(\frac{1-x}{x+1}\right), \quad e^{-d} = \frac{1-x}{x+1},$$

so,

$$xe^{-d} + e^{-d} = 1 - x, \quad x(e^{-d}+1) = 1 - e^{-d}, \quad x = \frac{1-e^{-d}}{1+e^{-d}}.$$

As a result we obtain:

$$x = \frac{e^{d/2} - e^{-d/2}}{e^{d/2} + e^{-d/2}}, \quad x = \tanh\left(\frac{d}{2}\right).$$

Lobachevskii function and unit of the length

The formula for the Lobachevskii function looks like:

$$\Pi(d) = \arccos\left(\tanh\left(\frac{d}{2}\right)\right)$$

and the inverse function is:

$$d = -\log\left(\frac{1-\cos(\alpha)}{1+\cos(\alpha)}\right) = -2\log\left(\tan\left(\frac{\alpha}{2}\right)\right), \quad 0 < \alpha < \pi/2.$$

This formula connects the length and angle. For the angle we have a natural measurement which is a radian. Therefore using this formula we can establish an universal unit for the measurement of length.
The universal unit for the length in the Lobachevskii geometry as $2\alpha = 1$ radian:

$$D = -2\log(\tan(1/4)) \sim 2.73030\ldots.$$

Pseudosphere

Coordinates on the surface of a unit sphere $x^2 + y^2 + z^2 = 1$:

$$x = \cos(\phi)\cos(\theta), \quad y = \sin(\phi)\cos(\theta), \quad z = \sin(\theta).$$

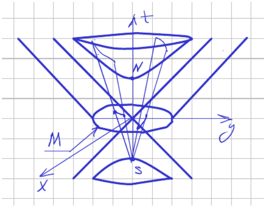
An equation for a pseudosphere in pseudo Euclid space with metrics $l^2 = t^2 - x^2 - y^2$ is a surface defined by equation $t^2 - x^2 - y^2 = 1$ and the parametric form:

$$t = \cosh(\chi), \quad x = \cos(\phi)\sinh(\chi), \quad y = \sin(\phi)\sinh(\chi),$$

The metrics on the pseudosphere:

$$(dl)^2 = (dt)^2 - (dx)^2 - (dy)^2 = -((d\chi)^2 + \sinh^2(\chi)(d\phi)^2).$$

Gnomonic projection



Define the coordinate on the unit circle as (u, v). The Gnomonic projection of the upper side hyperboloid onto the unit disc M.

$$\frac{x}{t+1} = \frac{t+1}{1}, \quad \frac{y}{t+1} = \frac{t+1}{1}.$$

Therefore:

$x = (t+1)u$, $y = (t+1)v$. The variable t depends on the variables (u, v):

$$t^2 - (t+1)^2u^2 - (1+t)^2v^2 = 1,$$

$$(1 - u^2 - v^2)t^2 - 2(u^2 + v^2)t - (1 + u^2 + v^2) = 0.$$

$$t = -1 + \frac{2}{1 - u^2 - v^2}.$$

The hyperboloid and the Klein's model

As a result the following formulas connect the surface of the hyperboloid and the points of the gnomonic projection:

$$x = (t+1)u,\ y = (t+1)v,\ t = -1 + \frac{2}{1-u^2-v^2}.$$

To connect a geodesic line on the hyperboloid and chord in the Klein's disc model for the Lobachevkii geometry one should make additional evaluations.

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