Klein's model of Lobachevskii geometry

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Bibliography

al-Haytham, Saccheri, Lamber

The Euclid's V postulate

- Al-Haytham about 1000 AD considered imaginary rectangles with one non-right angle. He tried to find contradiction an by this approach to prove V Euclid's postulate.
- Saccheri, whose work was published at 1733, tried to prove V Euclid's postulate using a contradict supposition. He prove few theorem in such way and assumed that he had found the contradictions.
- Later Lambert, whose work was published at 1786, understood the case with obtuse angle is connected to spherical geometry where the largest circles on the sphere considered instead of straight-lines.

Lobachavskii, Bolyai, Riemann

- Lobachavskii (1829) and Bolyai (1832) independently published their works concerning the non-Euclid geometry, where V postulate was changed by opposite one.
- Later at 1865 B.Riemann found unique definition on hyperbolic and spherical geometry.

Klein' model of Lobachevskii geometry

At 1871 year Felix Klein published work where mathematical model of the Lobachevskii geometry was proposal.
Let us consider an interior of a circle $\mathcal{U}$ such that $x^2 + y^2 = 1$ and two points $A(x_A, y_A)$ and $B(x_B, y_B)$ into this circle.

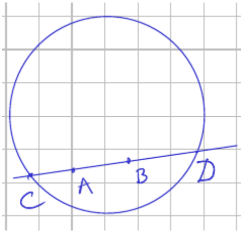
- A point in Klein's model of the Lobachevskii geometry is a point of interior of the unit circle.
- A straight line in the model is a chord of the unit circle.
- Axioms of a belonging and an order are the same as in Euclidian geometry.

A straight line in the Klein's model

- If $x_A \neq x_B$ and $y_A \neq y_B$, then these points are on the straight-line $\mathcal{L}$ is defined as follows $\frac{x-x_B}{x_A-x_B} = \frac{y-y_B}{y_A-y_B}$.
- If $x_A \neq x_B$ and $y_A = y_B$ then the straight-line $\mathcal{L}$ is $y = y_1$.
- If $x_A = x_B$ and $y_A \neq y_B$ then the straight-line $\mathcal{L}$ is $x = x_A$.

Define $\mathcal{L} \cap \mathcal{C} = \{C(x_C, y_C), D(x_D, y_D)\}$. For definitely let us assume $x_C < x_D$, $x_A \neq x_B$, and $y_C < y_D$ in opposite case.

A distance between two points



The distance between two different points is defined by follows rule:

$$|AB| = \left| \log \left(\frac{x_C - x_A}{x_C - x_B} : \frac{x_D - x_A}{x_D - x_B} \right) \right|, \quad x_A \neq x_B$$

$$|AB| = \left| \log \left(\frac{y_C - y_A}{y_C - y_B} : \frac{y_D - y_A}{y_D - y_B} \right) \right|, \quad y_A \neq y_B$$

here $C(x_C, y_C)$ and $D(x_D, y_D)$ are points on the circle $x^2 + y^2 = 1$ and the straight-line (A, B).

Klein's model of Lobachevskii geometry

Consider 3 measure axioms.

- It is easy to see next equalities:

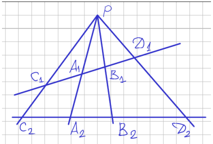
$$A \neq B \text{ then } |AB| > 0, \quad A = B \text{ then } |AB| = 0.$$

- Assume for definiteness that $x_A < x_B$. Let E be $E \in [A, B]$ and $[A, B] = [A, E] \cap [E, B]$

$$|AE| + |EB| = \log\left(\frac{x_C - x_A}{x_C - x_E} : \frac{x_D - x_A}{x_D - x_E}\right) +$$
$$\log\left(\frac{x_C - x_E}{x_C - x_B} : \frac{x_D - x_E}{x_D - x_B}\right) = \log\left(\frac{x_C - x_A}{x_C - x_B} : \frac{x_D - x_A}{x_D - x_B}\right) = |AB|.$$

- The triangle inequality, which are: if E be $E \notin [A, B]$, then $|A, E| + |E, B| > |A, B|$, will be proved below.

The cross ratio of four intervals



Let us consider two fractions. First one looks like:

$$\frac{|C_1A_1|}{|C_1B_1|} = \frac{S_{C_1A_1P}}{S_{C_1B_1P}} = \frac{|PC_1||PA_1|\sin(A_1PC_1)}{|PC_1||PB_1|\sin(B_1PC_1)} = \frac{|PA_1|\sin(A_1PC_1)}{|PB_1|\sin(B_1PC_1)},$$

and the same for another one fraction:

$$\frac{|D_1A_1|}{|D_1B_1|} = \frac{|PA_1|\sin(A_1PD_1)}{|PB_1|\sin(A_1PB_1)}.$$

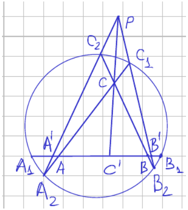
So, the following cross ration depends on the angles:

$$\frac{|C_1A_1|}{|C_1B_1|} : \frac{|D_1A_1|}{|D_1B_1|} = \frac{\sin(A_1PC_1)}{\sin(B_1PC_1)} : \frac{\sin(A_1PD_1)}{\sin(A_1PB_1)}.$$

Due to the equivalence of the angles on the vertex P:

$$\frac{|C_1A_1|}{|C_1B_1|} : \frac{|D_1A_1|}{|D_1B_1|} = \frac{|C_2A_2|}{|C_2B_2|} : \frac{|D_2A_2|}{|D_2B_2|}.$$

The triangle inequality



Due to the cross ratio:

$$\frac{|A_2A|}{|A_2C|} : \frac{|AC_1|}{|CC_1|} = \frac{|A'A|}{|A'C'|} : \frac{|B'A|}{|B'C'|}$$

and

$$\frac{|B_2B|}{|B_2C|} : \frac{|C_2B|}{|C_2C|} = \frac{|A'C'|}{|A'B|} : \frac{|B'C'|}{|B'B|}.$$

Now one should prove

$$\left| log\left(\frac{|A'A|}{|A'C'|} : \frac{|B'A|}{|B'C'|}\right)\right| + \left| log\left(\frac{|A'C'|}{|A'B|} : \frac{|B'C'|}{|B'B|}\right)\right| > \left|\log\left(\frac{|A_1A|}{|A_1B|} : \frac{|B_1A|}{|B_1B|}\right)\right|.$$

Transformations: rotation

Let us consider two different transformations of the circle.

- A rotation around of the origin is defined by following formulas:

$$x' = x\cos(\alpha) - y\sin(\alpha),$$
$$y' = x\sin(\alpha) + y\cos(\alpha).$$

  - Here we must notice that the formula for the distance independent on an angle, therefore the rotation concerns distance between two points.

Transformations: motion

The motion is defined by following formula:

$$x' = \frac{x\sqrt{1-\beta^2}}{1+\beta y}, \quad y' = \frac{y+\beta}{1+\beta y}, \quad \beta \in (-1,1).$$

The inverse transformation looks like:

$$x = \frac{x'\sqrt{1-\beta^2}}{1-\beta y'}, \quad y' = \frac{y-\beta}{1-\beta y'}.$$

The motion remains the sign of coordinate x, but change the scale and move the point $A(x,y)$ up or down with respect of the sing of parameter β.

Transformations: properties of the motion

- **The motion remains all points in the circle.** Indeed:

$$x'^2 + y'^2 - 1 = \frac{(1-\beta^2)(x^2+y^2-1)}{(1+\beta y)^2}.$$

  This expression is equal to zero if $x^2 + y^2 = 1$ and less than zero if $x^2 + y^2 < 1$.

- **The motion maps a straight line** $Ax + By = C$ **on straight line** $A'x' + By' = C'$**.**

Transformation: properties of the motion

- **The motion remains an order of points on a straight line.** To prove this let us consider a points on straight line $y = kx + c$ where $|c| < 1$. Then:

$$\frac{dx'}{dx} = \frac{d}{dx}\frac{x\sqrt{1-\beta^2}}{(1+\beta(kx+c)} = \frac{(c\beta+1)\sqrt{1-\beta^2}}{(kx\beta+c\beta+1)^2} > 0.$$

So the order does not change for the coordinate x. The same we can show for the transformation on the y coordinate:

$$\frac{dy'}{dy} = \frac{1-\beta^2}{(1+\beta y)^2} > 0.$$

Transformation: properties of the motion

- **The motion remains the distance between every two points.** To prove this one should use the formula for x' where y and x are defined by the straight line (A, B), like $y = kx + c$, and formula for the distance.

Angular measure

Let us postulate the angular measure of every angle ABC with a vertex B are the same for the the angular measure of an central angle. $A'OC'$ which are obtained by rotation and motion of the angle ABC.

The triangle $A(0,1/2)$, $B(-1/2,0)$, $O(0,0)$

The angular value $AOB = \pi/2$. To find the angular value of OAB one should move the point $A(0,1/2)$ into $(0,0)$ using the motion:

$$y' = \frac{y - 1/2}{1 - y/2}, \quad x' = \frac{x\sqrt{1 - 1/4}}{1 - y/2}.$$

So, $A'(0,0)$, $B'\left(-\sqrt{3}/4, -1/2\right)$, $O'(0,-1/2)$. Then the angular value of $O'A'B'$:

$$\arctan\left(\frac{\sqrt{3}}{2}\right) < \pi/4.$$

The angular value for $O'B'A'$ is equal to angular value of $O'A'B'$. As result we get the sum of angular values of vertexes for the triangle A, B, O is less than π.

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