Non-convergent and asymptotic series

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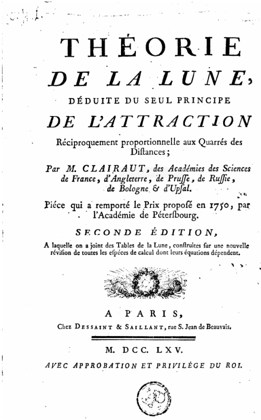
Zeno (V century BC)

The known aporia about Achilles and a turtle. If the turtle is behind Achilles then he do not catch it. Generally we consider two different problems.

1. First one is a convergence of time series to some value.
2. Second one is about representation of the time interval like a lengthened object and possibility to change it on a sequence of points, which do not have a length.

Figure: Zeno shows to his students doors to Truth and Lay. Fresco in Escorial Library, Spain

Euler and Clairaut 1747-1754



THÉORIE
DE LA LUNE,
DÉDUITE DU SEUL PRINCIPE
DE L'ATTRACTION
Réciproquement proportionnelle aux Quarrés des Diſtances;

Par M. CLAIRAUT, *des Académies des Sciences de France, d'Angleterre, de Pruſſe, de Ruſſie, de Bologne & d'Upſal.*

Piéce qui a remporté le Prix propoſé en 1750, par l'Académie de Péterſbourg.

SECONDE ÉDITION,

A laquelle on a joint des Tables de la Lune, conſtruites ſur une nouvelle réviſion de toutes les eſpéces de calcul dont leurs équations dépendent.

A PARIS,
Chez DESSAINT & SAILLANT, rue S. Jean de Beauvais.

M. DCC. LXV.
AVEC APPROBATION ET PRIVILÉGE DU ROI.

When one studied
a motion of a perigee of the Moon
it was found that this motion does not
obey to the universal law of gravitation.
However accuracy calculations
which consider a three bodies system
like the Sun, the Earth and the Moon,
which was made by Clairaut, show that
taking into account more terms of the
perturbation theory give a result which is
convince the universal gravitational law.
For these calculations Clairaut awarded
by special prize Russian academy of
Sciences.

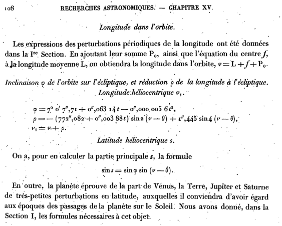
Leverrier and Adams 24.09.1846

- The planet Uranus was the first planet of the solar system discovered using a telescope. However, its motion does not obey exactly predicted by Newton's law of attraction.
- A well-known story is how the French mathematician Leverrier calculated the orbit of this hypothetical planet and predicted the area in the firmament in which it can be observed.

This story one can see in Scientific American, 2005, n3, p.52-59

Leverrier and Einstein from 1859 to 1915

The anomalous motion of Mercury's perihelion was known. Leverrier investigated the properties of motion perturbations. Up until 1915, there were false discoveries of planets responsible for these disturbances. In 1915, A. Einstein showed that the motion of the perihelion of Mercury can be explained within the framework of the general theory of relativity. (Einstein A. Explanation of the motion of the perihelion of Mercury in the general theory of relativity // Einstein's proceedings. volume 4. - T. I. - pp. 439-447.)



108 RECHERCHES ASTRONOMIQUES. — CHAPITRE XV.

Longitude dans l'orbite.

Les expressions des perturbations périodiques de la longitude ont été données dans la Ire Section. En ajoutant leur somme P_0, ainsi que l'équation du centre f, à la longitude moyenne L, on obtiendra la longitude dans l'orbite, $v = L + f + P_0$.

Inclinaison φ de l'orbite sur l'écliptique, et réduction ρ de la longitude à l'écliptique. Longitude héliocentrique v_1.

$$\varphi = 7^\circ\, 0'\, 7'',71 + 0'',063\,14\,t - 0'',000\,005\,6\,t^2,$$
$$\rho = -(772'',082 + 0'',003\,88\,t)\sin 2(v-\theta) + 1'',445 \sin 4(v-\theta),$$
$$v_1 = v + \rho.$$

Latitude héliocentrique s.

On a, pour en calculer la partie principale s, la formule

$$\sin s = \sin\varphi \sin(v-\theta).$$

En outre, la planète éprouve de la part de Vénus, la Terre, Jupiter et Saturne de très-petites perturbations en latitude, auxquelles il conviendra d'avoir égard aux époques des passages de la planète sur le Soleil. Nous avons donné, dans la Section I, les formules nécessaires à cet objet.

Figure: The picture from the work by Leverrier about a motion of the Mercury.

Quasi-classical approach 1926 -...

Relations between quantum mechanics and classical mechanics can be obtained, in particular, within the framework of the quasi-classical approximation.

- A rapidly oscillating solution of the Schrodinger equation is considered.
- The decomposition is constructed by a small parameter - the Planck constant.
- Within the framework of this approach, both the laws of linear optics and the equations of motion of Hamiltonian mechanics are obtained.

Synchronization from the pendulum to chaos

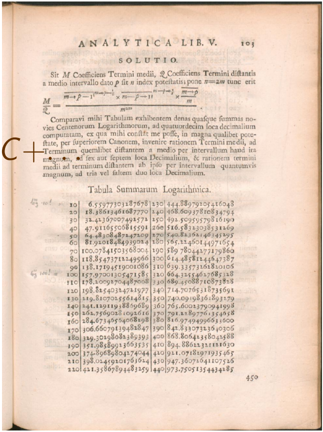
- Synchronization of the pendulum clock
- Synchronization of breathing in ventilation systems
- Synchronization of chaotic oscillations for hidden signal transmission

Moivre-Stirling formula, 1730

Moivre presented his work, in which, in particular, he found an approximate formula for the sum of logarithms

$$\sum_{k=1}^{N} \lg(k) \sim N\log(N) - N\log(e) + \frac{1}{2}\log(n) + C + \ldots$$

and presented calculations in the form of tables. Stirling in the same 1730 led the calculation of the constant with the Moivre formula. This constant turned out to be equal to $\sqrt{2\pi}$.



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SOLUTIO.

Sit M Coefficiens Termini medii, Q Coefficiens Termini diſtantis a medio intervallo dato p ſit n index poteſtatis; pone $n=2m$ tunc erit

$$\frac{M}{Q} = \frac{\overline{m+p-1}^{\,m+p-\frac{1}{2}} \times \overline{m-p+1}^{\,m-p+\frac{1}{2}} \times \frac{\overline{m+p}}{m}}{m^{2m}}.$$

Comparavi mihi Tabulam exhibentem denas quaſque ſummas novies Centenorum Logarithmorum, ad quatuordecim loca decimalium computatam, ex qua mihi conſtat me poſſe, in magna qualibet poteſtate, per ſuperiorem Canonem, invenire rationem Termini medii, ad Terminum quemlibet diſtantem a medio per intervallum haud ita magnum, ad ſex aut ſeptem loca Decimalium, & rationem termini medii ad terminum diſtantem ab ipſo per intervallum quantumvis magnum, ad tria vel ſaltem duo loca Decimalium.

Tabula Summarum Logarithmica.

10	6.55977303287678	230	444.88979205416048
20	18.38613461687770	240	468.60937810834794
30	32.42367007492572	250	492.50959579816190
40	47.91165506815591	260	516.58323038531269
50	64.48308487247209	270	540.82362146345295
60	81.92018484939024	280	565.22460144971654
70	100.07841503568004	290	589 78044277179860
80	118.85473712249966	300	614.48581244647387
90	138.17194519001086	310	639.33573262820106
100	157.97001305471585	320	664.32554627685328
110	178.20092704487008	330	689.45088710873828
120	198.82540324721977	340	714.70765318735691
130	219.81070255614815	350	740.09198361893279
140	241.12911938869689	360	765.60023790941998
150	262.75690281092616	370	791.22897761354658
160	284.67346564068298	380	816.97494996633600
170	306.66079139482847	390	842.83307323640306
180	329.30298082389393	400	868.80642358042588
190	351.98589923663535	410	894.88622321121630
200	374.89689804274044	420	921.07182971935465
210	398.02459201763624	430	947.36072641107526
220	421.35867894483259	440	973.75051354434285

450

Stirling series

In Stirling's formula, only the first term is usually calculated. However , in its full form , this formula contains a number of:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \exp\left(\sum_{k=1}^{\infty}\frac{B_{2k}}{2k(2k-1)n^{2k-1}}\right).$$

Here $B_{2k} \sim \frac{(2k)!}{(2\pi)^{2k}}$ is the the Bernoulli number and its growth rate at $k \to \infty$.

Bayes on the divergence of Stirling series (1763)

prefs any quantity at all; becaufe after the 5th term the coefficients begin to increafe, and they afterwards increafe at a greater rate than what can be compenfated by the increafe of the powers of z, though z reprefent a number ever fo large; as will be evident by confidering the following manner in which the coefficients of that feries may be formed. Take $a = \frac{1}{12}$, $5b = a^2$, $7c = 2ba$, $9d = 2ca + b^2$, $11e = 2da + 2cb$, $13f = 2ea + 2db + c^2$, $15g = 2fa + 2eb + 2dc$, and fo on; then take $A = a$, $B = 2b$, $C = 2 \times 3 \times 4c$, $D = 2 \times 3 \times 4 \times 5 \times 6d$, $E = 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8e$ and fo on, and A, B, C, D, E, F, &c. will be the coefficients of the foregoing feries: from whence it eafily follows, that if any term in the feries after the 3 firft be called y, and its diftance from the firft term n, the next term immediately following will be greater than $\frac{n \times \overline{2n-1}}{6n+9} \times \frac{y}{z^2}$. Wherefore at length the fubfequent terms of this feries are greater than the preceding ones, and increafe in infinitum, and therefore the whole feries can have no ultimate value whatfoever.

Bayes in 1763 noticed
that the series in Stirling's formula diverges, although the first few terms decrease one after another, so that their calculation leads to an improvement in the approximation. However, the full series diverges.

Euler's solution for DU

$$\phi(x) \sim x - x^2 + 2!x^3 - \cdots + (-1)^{n-1}(n-1)!x^n + \ldots .$$

$$\begin{aligned} x^2\phi' + \phi &= x^2(1 - 2!x + 3!x^2 + \cdots + (-1)^{n-1}n!x^{n-1} + \ldots) + \\ & (x - x^2 + 2!x^3 + \cdots + (-1)^n n!x^{n+1} + \ldots) = x. \end{aligned}$$

Hence:

$$x^2\phi' + \phi = x. \quad \phi = ce^{1/x} + e^{1/x}\int_0^x e^{-1/t}\frac{dt}{t}.$$

$$t = \frac{x}{1+wx}, \quad \frac{\phi}{x} = \int_0^\infty \frac{e^{-w}dw}{1+xw}.$$

From Abel (1828) to Poincare (1892)

Abel (1828) about divergent series: "It is shameful to base any proof on them" (quoted by Ramis J.P. Divergent series and asymptotic theories).
Poincare: "consider two series whose common term has the form:

$$\frac{1000^n}{n!} \quad \text{and} \quad \frac{n!}{1000^n}.$$

Geometers will say that the first one diverges, and quickly ... Astronomers, on the contrary, will be the first divergent ..., and the second converging." (A. Poincare, New Methods of Celestial Mechanics, Vol. 2).

Definition of Poincare

Consider a divergent series:

$$\sum_{k=0}^{\infty} f_k(x)\mu^k,$$

denote

$$\phi_p(x,\mu) = \sum_{k=0}^{p} f_k(x)\mu^k.$$

If the condition is met:

$$\lim_{\mu\to 0} \frac{\phi(x, mu) - \phi_p(x,\mu)}{\mu^p} = 0$$

then the series is an asymptotic representation of the function

$$\phi(x,\mu) \sim \sum_{k=0}^{\infty} f_k(x)\mu^k.$$

The existence of a function with a given asymptotic expansion

Consider the series

$$f_0 + f_1 x + \sum_{k=2}^{\infty} f_k x^k \left(1 - \exp\left(-\frac{1}{2^k |f_k| x}\right)\right). \qquad (1)$$

- The series (1) at $x \to +0$ is asymptotically equivalent to the series $f_0 + f_1 x + f_2 x^2 + \ldots$.
- Row (1) converges for $0 < x < 2$, since:

$$\left| \sum_{k=2}^{\infty} f_k x^k \left(1 - \exp\left(-\frac{1}{2^k |f_k| x}\right)\right) \right| <$$

$$\sum_{k=2}^{\infty} |f_k| \frac{x^k}{2^k |f_k| x} = \sum_{k=2}^{\infty} \frac{x^{k-1}}{2^k}.$$

The asymptotics of the integral, an example from the book by F.Olver

$$\int_0^\infty \frac{e^{-xt}}{1+t}dt = \int_0^\infty e^{-xt}\left(1 - t + t^2 + \dots + (-1)^n t^n + \dots\right) dt =$$
$$\frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \dots + (-1)^{n-1}\frac{(n-1)!}{x^n} + \dots .$$

The series diverges, but if we take $x = 10$ and calculate the value of the integral numerically: 0.09156 and from the resulting formula by four terms:

$$0.1 - 0.01 + 0.002 - 0.0006 + 0.00024 - 0.00012 = 0.09152$$

The answer is close to the correct one.

An example from the book by F.Olver

$$\int_0^\infty \frac{e^{-xt}}{1+t}dt = \int_0^\infty e^{-xt}\left(1 - t + t^2 + \cdots + (-1)^n \frac{t^n}{1+t}\right)dt =$$
$$\frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} + \cdots + (-1)^{n-1}\frac{(n-1)!}{x^n} + (-1)^n \int_0^\infty \frac{t^n e^{-xt}}{1+t}dt$$

$$\int_0^\infty \frac{t^n e^{-xt}}{1+t}dt < \int_0^\infty \frac{t^n e^{-xt}}{1+t}dt = \frac{n!}{x^{n+1}}.$$

We can use a segment of a series of such length, as long as each subsequent member of the series is smaller than the previous one.

The asymptotics of the sum of the series. Example of A.M.Ilyin and A.R.Danilin

Calculate the sum of the series with an accuracy of three digits:

$$\sum_{k=2}^{\infty} \frac{1}{n \ln(n)}$$

It follows from the Cauchy integral sign that the series converges. Let's estimate the remainder of the row from below:

$$\sum_{k=m}^{\infty} \frac{1}{n \ln(n)} > \int_{m+1}^{\infty} \frac{dn}{n \ln(n)} = \frac{1}{\ln(m+1)}$$

That is, for three significant digits, you need $\ln(m+1) \sin 10^3$ or $m > e^{1000} > 10^{300}$. The age of the universe is estimated as 10^{18}sec.

The asymptotics of the sum of the series. Example of A.M.Ilyin and A.R.Danilin

$$\sum_{k=2}^{\infty} \frac{1}{n \ln(n)} \sim \sum_{k=2}^{m} \frac{1}{n \ln(n)} + \frac{1}{\ln(m+1/2)} -$$

$$\frac{1}{24}\left(\frac{1}{(m+1/2)^2 \ln^2(m+1/2)} + \frac{2}{(m+1/2)^2 \ln^3(m+1/2)}\right)$$

The error is estimated as follows $(1/2)m^{-2}$.

Fresnel's integral

$$I(x) = \int_x^\infty \sin(y^2)dy. \tag{2}$$

$$\int_x^\infty \sin(y^2)dy = -\int_x^\infty \frac{d\cos(y^2)}{2y} = \frac{\cos(x^2)}{2x} - \int_x^\infty \frac{d\sin(y^2)}{4y^3} = \frac{\cos(x^2)}{2x} + \frac{\sin(x^2)}{4x^3} + \int_x^\infty \frac{3d\cos(y^2)}{8y^5}.$$

$$\int_x^{\infty} \sin(y^2)dy \sim \frac{\cos(x^2)}{2x}\sum_{n=0}^{\infty}(-1)^n\frac{3\times 5\times 7\times\cdots\times(4n-1)}{2^{2n}*x^{4n}}+$$

$$\frac{\sin(x^2)}{4x^3}\sum_{n=0}^{\infty}(-1)^n\frac{3\times 5\times 7\times\cdots\times(4n+1)}{2^{2n}x^{4n}}. \quad (3)$$

$$\frac{a_{n+1}}{a_n}=-\frac{(4n+1)(4n+3)}{4x^4}=0, \quad x\to\infty;$$

$$\frac{b_{n+1}}{b_n}=-\frac{(4n+5)(4n+7)}{4x^4}=0, \quad x\to\infty;$$

$$\int_x^\infty \sin(y^2)dy =$$

$$\frac{\cos(x^2)}{2x}\sum_{n=0}^{N}(-1)^N\frac{3\times 5\times 7\times\cdots\times(4n-1)}{2^{2n}x^{4n}}+$$

$$\frac{\sin(x^2)}{4x^3}\sum_{n=0}^{N}(-1)^N\frac{3\times 5\times 7\times\cdots\times(4n+1)}{2^{2n}x^{4n}}+$$

$$+O(x^{-4N-2}),$$

$$\int_x^{\infty} \sin(y^2)dy =$$

$$\sqrt{\frac{\pi}{2}} - \sum_{n=0}^{\infty} \frac{(-1)^n \, x^{4n+3}}{(4n+3) \times (2n+1)!}. \qquad (4)$$

To calculate the Fresnel integral at $x = 10$ with an accuracy of 0.01, more than 135 terms of the series are needed:

$$\frac{10^{4N+3}}{(4N+3) \times (2N+1)!} \leq 10^{-2}$$

or

$$10^{4N+5} \leq (4N+3) \times (2N+1)!$$

$$\log(4N+3) + \sum_{k=1}^{2N+1} \log(k) - (4N+5)\log(10) \geq 1.$$

$N > 135.$

Riemann's theorem on summation of conditionally convergent series

Let the series $\sum_{k=1}^{\infty} a_k$ conditionally converges. Then $\forall Z$ $\exists \sigma(k)$: $\sum_{k=1}^{\infty} a_{\sigma(k)} = Z$.
An example. $a_k = (-1)^k/k$. Let us show that one can construct $\sigma(k)$ for $Z = 1.5$:

$$1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15} - \frac{1}{4} + \cdots = \\ 1.5(3) - 0.5 + 0.4(884670) - 0.25 + \ldots .$$

Summation

by Chezaro:

$$S = \lim_{n\to\infty} \frac{1}{n} \sum_{n+1}^{n} s_n, \quad s_n = \sum_{k=1}^{n} a_k.$$

By Borel:

$$S = \int_0^\infty e^{-t} \sum_{n=0}^{\infty} \frac{t^n}{n!} a_n, \quad A(z) = \sum_{k=0}^{\infty} a_k z^k.$$

Example

Let us consider a divergent series

$$S = \sum_{n=0}^{\infty} (-1)^n.$$

The sum of this series by Chezaro's rule:

$$S = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{j=0}^{k} (-1)^j = \lim_{2n \to \infty} \frac{n}{2n} = \frac{1}{2}.$$

The sum of this series by the Borel's rule:

$$S = \int_0^{\infty} e^{-t} \sum_{n=0}^{\infty} \frac{t^n}{n!} (-1)^n dt = \int_0^{\infty} e^{-2t} dt = \frac{1}{2}.$$

Euler's example

Euler's example:

$$A(z) = \sum_{n=0}^{\infty} n!(-1)^n z^n$$

. This series does not converge by Chezaro, but converges by Borel's rule for summation:

$$\int_0^{\infty} e^{-t} \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{n!} n!(-1)^n z^n dt =$$

$$\int_0^{\infty} e^{-t} \sum_{n=0}^{\infty} (tz)^n dt = \int_0^{\infty} \frac{e^{-t} dt}{1 + tz}.$$

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