Triple integrals

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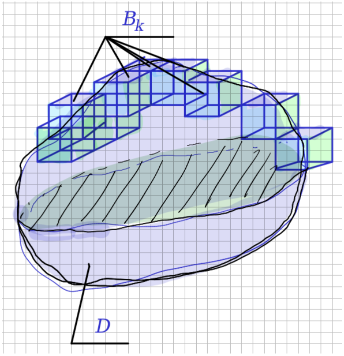
Changing variables

Definition of measure for volume of 3-dimensional body



A subset $\mathcal{D}$ of $\mathbb{R}^3$ has a measurable 3D measure if there exists a non-negative real number V such that $\forall \epsilon > 0$, $\exists \cup_{k=1}^{n} B_k$ of rectangular boxes B_k such that $\mathcal{D} \subset \bigcup_{i=1}^{n} B_i$ and

$$\sum_{i=1}^{n} |B_i| < V + \epsilon,$$

where $|B_i|$ denotes the volume of the rectangular box B_i. The number V is called the 3D volume

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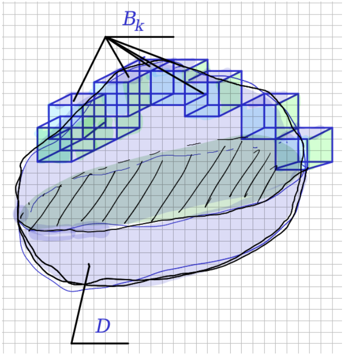

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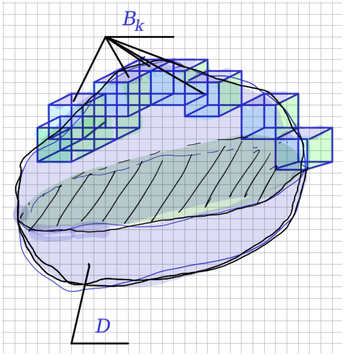

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A Darboux criteria for the existence of the measure for given 3D body

A subset $\mathcal{D}$ of $\mathbb{R}^3$ is measurable if and only if $\forall \epsilon > 0$, $\exists A = \cup_{k=1}^{n} A_k$ and $B = \cup_{k=1}^{n} B_k$ where A_k and B_k are rectangular boxes, $A \subset \mathcal{D} \subset B$ and

$$|B \setminus A| < \epsilon,$$

where $|B \setminus A|$ denotes the 3D volume of the set difference $B \setminus A$.

In other words, a subset $\mathcal{D}$ of $\mathbb{R}^3$ is measurable if and only if it can be enclosed by two finite unions of rectangular boxes with arbitrarily close volumes.

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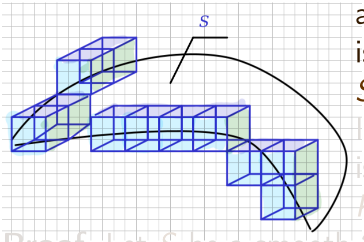
A theorem about measurable of bounded 3d body with measurable border surface



Let $S = \partial\mathcal{D}$ be a smooth and bounded surface in $\mathbb{R}^3$. If S is measurable as 2D surface, then S has zero volume (3D measure).

In other words, if S is a smooth 2D surface in compact $K \in \mathbb{R}^3$, then its area is zero.

Proof. Let S be a smooth 2D surface in $\mathbb{R}^3$. Without loss of generality, we can assume that S is contained in a compact set K in $\mathbb{R}^3$. Let U_i be a countable covering of K by open balls such that the diameter of each ball is less than ϵ, where $\epsilon > 0$ is fixed. Let $S_i = S \cap U_i$ be the intersection of S with each ball U_i.

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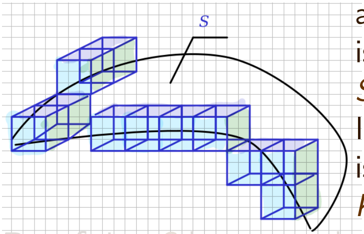

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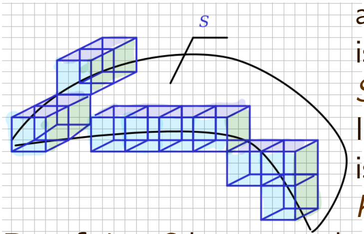

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Proof of the theorem about volume of smooth surface.

Now, we have:

$$S \subset \bigcup_i S_i \subset \bigcup_i (D_i \times [0, \epsilon])$$

where $[0, \epsilon]$ is the interval of length ϵ in the z-direction. Therefore, we have:

$$\text{vol}(S) \leq \sum_i \text{vol}(D_i \times [0, \epsilon]) = \epsilon \sum_i \text{vol}(D_i)$$

where $\text{vol}(D_i)$ denotes the area of the compact set D_i. Since S is contained in the compact set K, we have $\sum_i \text{vol}(D_i) \leq \text{vol}(K) < \infty$.

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Counterexample. The Mundelbulb 3D fractal

Stolen from Internet!

Definition of triple integral

Let D be a bounded and measurable domain in $\mathbb{R}^3$ and let $f : D \to \mathbb{R}$ be a function. The **triple integral** of f over D is denoted by

$$\iiint_D f(x, y, z)\, dx\, dy\, dz,$$

and is defined as the limit of Riemann sums as the mesh size approaches zero:

$$\iiint_D f(x, y, z)\, dx\, dy\, dz = \lim_{\max(\Delta V_{ijk}) \to 0} \sum_{i,j,k} f(\xi_{ijk}, \eta_{ijk}, \zeta_{ijk})\, \Delta V_{ijk},$$

where $(\xi_{ijk}, \eta_{ijk}, \zeta_{ijk}) \in V_{ijk}$, and $\Delta V_{ijk} = vol(V_{ijk})$.

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Theorem about existence of triple integrals

(Existence of Triple Integral) Let D be a bounded and measurable domain in $\mathbb{R}^3$, and let $f : D \to \mathbb{R}$ be a function. If f is continuous on D, then the triple integral $\iiint_D f(x, y, z)\, dx\, dy\, dz$ exists.

A proof of the existence theorem

Proof: We will prove the existence of the triple Riemannian integral by showing that it can be approximated by a sequence of triple Riemann sums.

Let $P = \{(x_0, y_0, z_0), (x_1, y_1, z_1), \ldots, (x_n, y_n, z_n)\}$ be a partition of D into n subdomains, such that each subdomain is a rectangular box. Let $\Delta x_i = x_i - x_{i-1}$, $\Delta y_j = y_j - y_{j-1}$, and $\Delta z_k = z_k - z_{k-1}$ be the lengths of the sides of the boxes. Then the volume of each subdomain is $\Delta V_{ijk} = \Delta x_i \Delta y_j \Delta z_k$. Let $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ be a point in the ijk-th subdomain. Then the triple Riemann sum for $f(x, y, z)$ over the partition P is:

$$S(P, f) = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\Delta V_{ijk}$$

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By the continuity of f, we can choose the points $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$ in such a way that the difference between $S(P, f)$ and the triple Riemannian integral is arbitrarily small:

$$\left| \iiint_D f(x, y, z) dV - S(P, f) \right| < \epsilon$$

for any $\epsilon > 0$.

Therefore, by the definition of the limit, the triple Riemannian integral $\iiint_D f(x, y, z) dV$ exists, and is equal to the limit of the sequence of triple Riemann sums as the size of the partition approaches zero.

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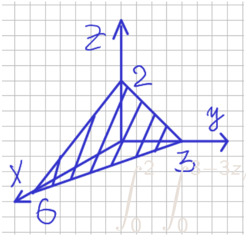
Example of a triple integral



$$\iiint_{\substack{z>0,y>0,x>0,\\ x+2y+3z-6=0}} (x+y+z)dxdydz =$$

$$\int_0^2\int_0^{3-3z/2}\int_0^{6-2y-3z}(x+y+z)dxdydz=$$

$$\int_0^2\int_0^{3-3z/2}\left(\frac{3z^2}{2}+yz-12z-6y+18\right)dydz$$

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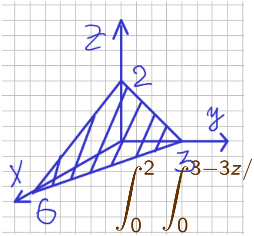

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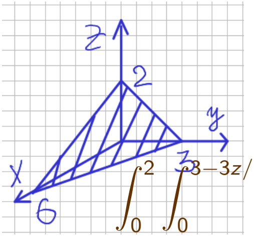

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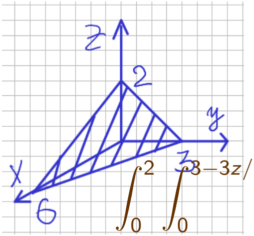

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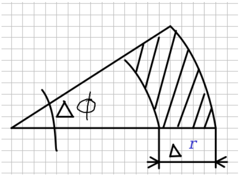
From Cartesian to polar coordinate system



Consider
an elementary plate of the area
on the plane in a polar coordinates.

$$ds = r\, dr\, d\phi.$$

Consider an integral over an area
with rectifiable border:

$$\iint_{\mathcal{D}} dxdy = \iint_{\mathcal{D}} ds = \iint_{\mathcal{D}} rdrd\phi$$

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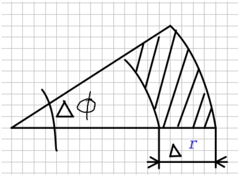

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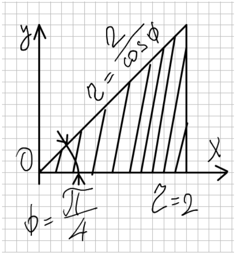

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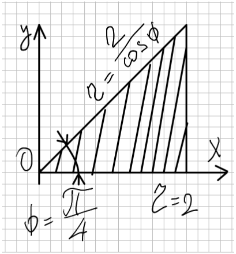

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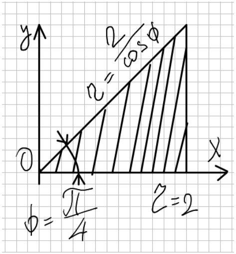

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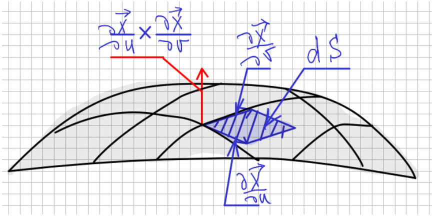
Changing of variables in two dimensional integrals

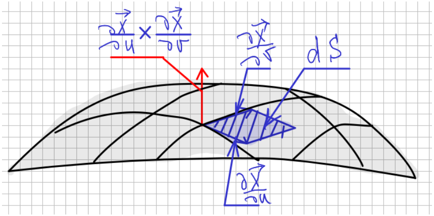


Let's consider a smooth surface S defined by a parametrization $\vec{\mathbf{x}} = \vec{\mathbf{x}}(u, v)$, where (u, v) are parameters in some domain $D \subset \mathbb{R}^2$. The elementary area of S at a point $\vec{\mathbf{x}}(u_0, v_0)$ is given by:

$$dS = \left\|\frac{\partial \vec{\mathbf{x}}}{\partial u} \times \frac{\partial \vec{\mathbf{x}}}{\partial v}\right\| du\, dv$$

where $\| \cdot \|$ denotes the Euclidean norm, and $\frac{\partial \vec{\mathbf{x}}}{\partial u}$, $\frac{\partial \vec{\mathbf{x}}}{\partial v}$ are the partial derivatives of $\vec{\mathbf{x}}$ with respect to u and v, respectively.

Changing of variables in two dimensional integrals



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$$dS = \| \frac{\partial \vec{\mathbf{x}}}{\partial u} \times \frac{\partial \vec{\mathbf{x}}}{\partial v} \| du\, dv$$

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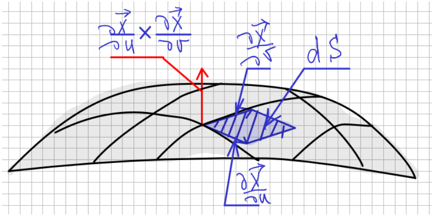

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Now, suppose we have a change of variables $(u, v) = (u(r, s), v(r, s))$. Let $\vec{\mathbf{y}} = \vec{\mathbf{x}}(u(r, s), v(r, s))$ be a new parametrization of the surface S in terms of the new variables (r, s). Then the partial derivatives of $\vec{\mathbf{y}}$ with respect to r and s are given by the chain rule:

$$\frac{\partial \vec{\mathbf{y}}}{\partial r} = \frac{\partial \vec{\mathbf{x}}}{\partial u}\frac{\partial u}{\partial r} + \frac{\partial \vec{\mathbf{x}}}{\partial v}\frac{\partial v}{\partial r}$$

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Changing of variables in two dimensional integrals

Taking the cross product of these vectors, we have:

$$\frac{\partial \vec{\mathbf{y}}}{\partial r} \times \frac{\partial \vec{\mathbf{y}}}{\partial s} = \left(\frac{\partial u}{\partial r}\frac{\partial v}{\partial s} - \frac{\partial u}{\partial s}\frac{\partial v}{\partial r}\right)\frac{\partial \vec{\mathbf{x}}}{\partial u} \times \frac{\partial \vec{\mathbf{x}}}{\partial v}$$

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Changing of variables in two dimensional integrals

Therefore, the elementary area changes by a factor of $\left|\frac{\partial u}{\partial r}\frac{\partial v}{\partial s} - \frac{\partial u}{\partial s}\frac{\partial v}{\partial r}\right|$ when changing variables from (u, v) to (r, s).

This is known as the Jacobian determinant of the change of variables, and it appears in many areas of mathematics, including multivariable calculus and differential geometry.

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Changing Cartesian to polar coordinate from general point of view

The integral in Cartesian coordinates is given by:

$$\int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x, y)\, dx\, dy$$

We'll change the variables to polar coordinates, where $x = r\cos(\theta)$ and $y = r\sin(\theta)$. When we make this transformation, the differential area element in polar coordinates, ds, is given by:

$$ds = r\, dr\, d\theta$$

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$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix} = r\cos^2(\theta) + r\sin^2(\theta) = r$$

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Changing of variables in triple integrals

Consider a function $f(x, y, z)$ defined on a region D in three-dimensional space, and express the integral of f over D in terms of a new set of coordinates (u, v, w), where $x = x(u, v, w)$, $y = y(u, v, w)$, and $z = z(u, v, w)$. Then the triple integral can be written as:

$$\iiint_D f(x, y, z) dV =$$

$$\iiint_{D'} f(x(u, v, w), y(u, v, w), z(u, v, w)) |J(u, v, w)| dudvdw,$$

where D' is the region in the u, v, w coordinate system that corresponds to the region D in the x, y, z coordinate system, and $J(u, v, w)$ is the Jacobian determinant of the transformation.

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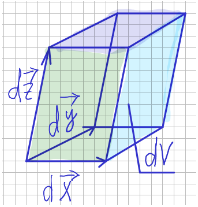

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The Jacobian measures the change in volume due to the change of variables.

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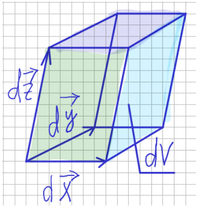

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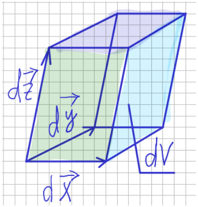

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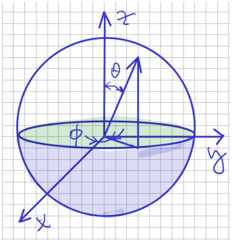

Let a function $f(x, y, z)$ be defined in Cartesian coordinates (x, y, z). Change to spherical coordinates (r, θ, ϕ), where r is the radial distance from the origin, θ is the polar angle measured from the positive z-axis, and ϕ is the azimuthal angle measured from the positive x-axis in the xy-plane.

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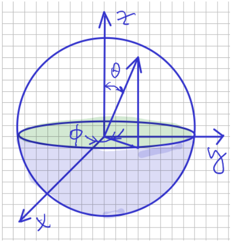

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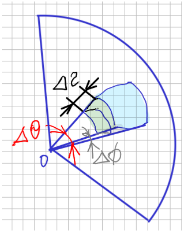

The transformation
from Cartesian coordinates
to spherical coordinates is given by:

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$$

where
$0 \le r \le R$, $0 \le \theta \le \pi$, and $0 \le \phi \le 2\pi$.

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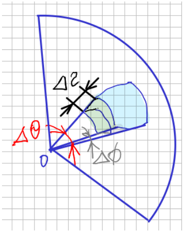

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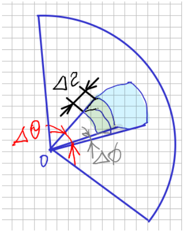

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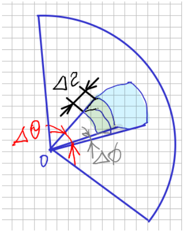

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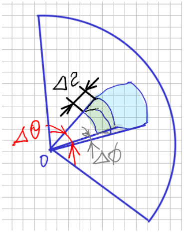

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Therefore, the integral of f over the sphere of radius R centered at the origin in spherical coordinates is given by:

$$\int_0^{2\pi} \int_0^{\pi} \int_0^{R} f(r \sin\theta \cos\phi, r \sin\theta \sin\phi, r \cos\theta)\, r^2 \sin\theta \, dr\, d\theta\, d\phi$$

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$$\iiint_{x^2+y^2+z^2<16} \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz =$$

$$\int_0^{2\pi}\int_{-\pi/2}^{\pi/2}\int_0^4 r^3\cos(\psi)dr\,d\psi\,d\phi = \int_0^{2\pi}\int_{-\pi/2}^{\pi/2} 64\cos(\psi)\,d\psi\,d\phi =$$

$$\int_0^{2\pi} 128 d\psi = 256\pi.$$

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$$\iiint_{x^2+y^2+z^2<4} \sqrt{x^2+y^2+z^2}dx\,dy\,dz.$$

$$\iiint_{x^2+y^2+z^2<16} \sqrt{x^2+y^2+z^2}dx\,dy\,dz =$$

$$\int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \int_0^4 r^3 \cos(\psi) dr\,d\psi\,d\phi = \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} 64 \cos(\psi)\,d\psi\,d\phi =$$

$$\int_0^{2\pi} 128 d\psi = 256\pi.$$

Spherical coordinates

$$\iiint_{x^2+y^2+z^2<4} \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz.$$

$$\iiint_{x^2+y^2+z^2<16} \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz =$$

$$\int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \int_0^4 r^3 \cos(\psi) dr\,d\psi\,d\phi = \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} 64 \cos(\psi)\,d\psi\,d\phi =$$

$$\int_0^{2\pi} 128 d\psi = 256\pi.$$