Multiple integrals

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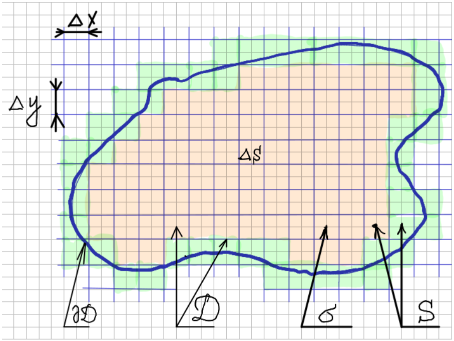
External integral sum

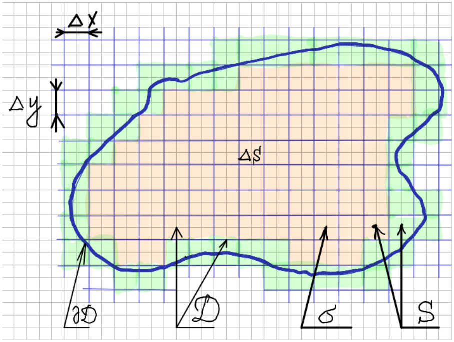


Consider an area $\mathcal{D}$ on the plane. Divide the area on a mesh with steps Δx and Δy. rectangle element of the plane $\Delta s = \Delta x \Delta y$. Cover the $\mathcal{D}$ by the rectangles $\Delta s = \Delta x \Delta y$ the and define the sum of the rectangles, which cover the $\mathcal{D}$:

$$S = \sum_{N} \Delta s.$$

Here N is the number of the elements Δs which covered the area $\mathcal{D}$.

Internal integral sum



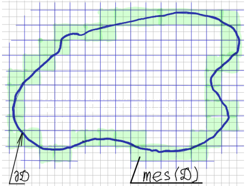
Define σ as a sum of the rectangles Δs which are internal of the $\mathcal{D}$:

$$\sigma = \sum_M \Delta s.$$

Here M is number of the internal rectangles for the $\mathcal{D}$, $M \leq N$ Then the area of the figure $\mathcal{D}$:

$$\sum_M \Delta s \leq \mathrm{mes}\mathcal{D} \leq \sum_N \Delta s.$$

An area of the border



Define a difference between sum external and internal rectangles as a area of the border:

$$\mathrm{mes}(\partial\mathcal{D}) \leq (N - M)\Delta s.$$

Theorem. **A measure of a rectifiable curve is equal to zero.**

Proof. Let the length of the curve $\mathcal{L}$ be equal to l. Divide the curve over n segments with the same length. Then any segment of the curve can be covered by a circle of diameter l/N. The measure of all such circles are

$$\mathrm{mes}(\mathcal{L}) \leq \lim_{N\to\infty} \sum_{i=1}^{N} \left(\frac{l}{N}\right)^2 \pi \leq \lim_{N\to\infty} \frac{l^2}{N} = 0.$$

Theorem about a measure of the set

If a border $\partial\mathcal{D}$ of a certain area $\mathcal{D}$ is rectifiable curve, then the area is measurable.

Proof. Define a rectangle $\Delta s = \Delta x \Delta y$ and the set of the rectangles with the area Δs $s_i \subset \mathcal{D}$ and $\mathrm{mes}(\cap s_i) = 0$.

$$\sigma_M = \sum_M \Delta s.$$

Define a mesh which covers the set $\mathcal{D}$:

$$\{s_i\},\ s_i \cap \mathcal{D} \neq 0,\ s_N = \sum_N \Delta s,\ \sigma_M \leq s_N,\ M < N.$$

The limit as $\max\{\Delta x, \Delta y\} \to 0$, then $M, N \to \infty$ and

$$\mathrm{mes}(\partial\mathcal{D}) = 0, \Rightarrow s_N - \sigma_M \to 0,\ \Rightarrow$$

$$\sigma_M \leq \mathrm{mes}(\mathcal{D}) \leq s_N \Rightarrow \mathrm{mes}(\mathcal{D}) = \lim_{\Delta s \to 0} \sigma_M = \lim_{\Delta s \to 0} s_N.$$

Counter example. The area of the Koch snowflake

Find the area of the Koch snowflake if the initial length of the side of the triangle was 1.

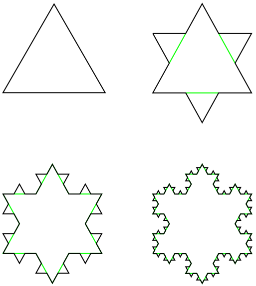


Figure: Koch snowflake. The picture from Wikipedia

The perimeter of the Koch snowflake: Number of sides $N_n = 3 \cdot 4^n$, the length of the side $l_n = 3^{-n}$, perimeter is equal

$$P = N_n \cdot L = 3 \left(\frac{4}{3}\right)^n,$$

The area of the Koch's snowflake

The length of each interval on the n-th step $l_n = 1/3^n$, numbers of net triangles $N_n = 3 \cdot 4^{n-1}$ and every step add the triangle with area $s_n = s_{n-1}/9$. The area of the initial triangle $s_0 = \sqrt{3}/4$. Therefore

$$\begin{aligned}
S_n &= s_0 \left(1 + \sum_{n=1}^{\infty} 3 \cdot 4^{n-1} \cdot \frac{s_0}{9^n} \right) = s_0 \left(1 + \frac{3}{4} \sum_{n=1}^{\infty} \left(\frac{4}{9} \right)^n \right) = \\
&= s_0 \left(1 + \frac{3}{4} \left(\sum_{n=0}^{\infty} \left(\frac{4}{9} \right)^n - 1 \right) \right) = s_0 \left(1 + \frac{3}{4} \left(\frac{1}{1 - \frac{4}{9}} - 1 \right) \right) \\
&= s_0 \left(1 + \frac{3}{4} \left(\frac{9}{5} - 1 \right) \right) = s_0 \frac{8}{5}. \\
&= \frac{2\sqrt{3}}{5}.
\end{aligned}$$

The Rimannian integral

Let the set $\mathcal{D}$ has a rectifiable border $\partial\mathcal{D}$ and a continuous function $f(x, y)$ is defined over all $\mathcal{D}$. Any sum

$$I = \lim_{\max(\Delta x, \Delta y)\to 0} \sum_{n=1}^{N} f(x_i, y_i)\Delta s_i$$

is called Rimanian integral of the function $f(x, y)$ over the set $\mathcal{D}$.

The integral is written as follows:

$$I = \int_{\mathcal{D}} f(x, y)ds \equiv \int\int_{\mathcal{D}} f(x, y)dxdy.$$

Theorem about existence of the double integral

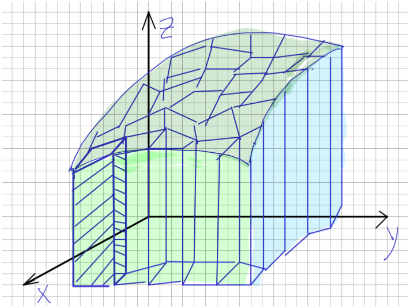
If the set $\mathcal{D}$ has a rectifiable border $\partial\mathcal{D}$ and a continuous function $f(x,y)$ is defined over all $\mathcal{D}$ then the Rimannian integral exists.

Sketch of a proof. Let's consider a mesh of measurable parts δs_i of the set $\mathcal{D}$ and define the upper limit F_i of $f(x,y)$ on Δs_i and lower limit of $f(x,y)$ on the Δs_i.
The Rimannian integral then one obtains an inequalities:

$$\sum_{i=1}^{N} f_i \Delta s_i \leq \sum_{n=1}^{N} f(x_i, y_i)\Delta s_i \leq \sum_{i=1}^{N} F_i \Delta s_i.$$

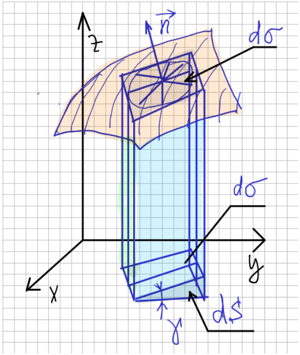
Define $\mathrm{diam}(s_i) = \sup_{\{A,B\}\in s_i}(\mathrm{dist}(A,B))$.
When $\mathrm{diam}(\Delta s_i) \to 0$ the difference $F_i - f_i \to 0$ due to continuity $f(x,y)$. Hence the Rimanian integra exists and the value of the does not depends of the point $(x_i, y_i) \in s_i$.

Geometrical sense of the double integral



The double integral of function $f(x, y)$ might be considered as a volume between the area $\mathcal{D}$ and the surface $f(x, y)$.

Geometrical sense of the double integral



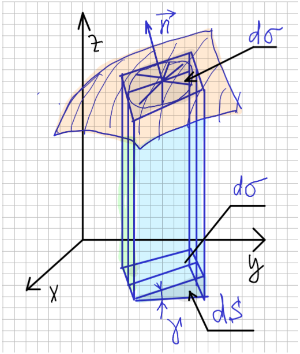
Let's consider
an elementary area on a surface $z = f(x, y)$. Suppose a projection of the $d\sigma$ on the plane xOy is the area ds. The normal of the surface at the point (x, y, z) is follows:

$$\vec{\nabla} = \left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \right),$$

then unit normal vector:

$$\vec{n} = \frac{1}{\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}} \left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \right).$$

Geometrical sense of the double integral



The projection of $d\sigma$ and dS connected by a formula:

$$d\sigma \cdot \cos(\gamma) = dS \Rightarrow d\sigma = \frac{dS}{\cos(\gamma)},$$

define $\vec{e}_3 = (0, 0, 1)$, then

$$\cos(\gamma) = (\vec{n}, \vec{e}_3) = \frac{1}{\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}}.$$

$$\sigma = \iint_{\mathcal{D}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, ds.$$

Properties of the double integral

The sum of double integrals

Consider continuous function $f(x, y)$ on $\mathcal{D}$. Let $\mathrm{mes}(\partial\mathcal{D})$, $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$, $\mathcal{D}_1 \cap \mathcal{D}_2 = 0$ and $\mathrm{mes}(\partial\mathcal{D}_{1,2}) = 0$, then

$$\int_{\mathcal{D}} f(x, y)ds = \int_{\mathcal{D}_1} f(x, y)ds + \int_{\mathcal{D}_2} f(x, y)ds.$$

Sketch of proof. One can proof this property if one considers the integrals by definition and uses the properties of the borders.

Properties of the double integral

Estimation of the double integral

Let $S = \text{mes}(\mathcal{D})$ $f(x,y)$ is continuous function and $f = \min_{(x,y)\in\mathcal{D}} f(x,y)$, $F = \max_{(x,y)\in\mathcal{D}} f(x,y)$, then

$$S\,f \leq \int_{\mathcal{D}} f(x,y)ds \leq S\,F$$

Theorem about an average value

Let $S = \text{mes}(\mathcal{D})$, $f(x, y)$ is continuous function and $f = \min_{(x,y)\in\mathcal{D}} f(x, y)$, $F = \max_{(x,y)\in\mathcal{D}} f(x, y)$, then exists (x_m, y_m) such that:

$$\int_{\mathcal{D}} f(x, y)ds = Sf(x_m, y_m).$$

Proof.

$$S\,f \leq \int_{\mathcal{D}} f(x, y)ds \leq S\,F \Rightarrow f \leq \frac{1}{S}\int_{\mathcal{D}} f(x, y)ds \leq F,$$

then due to continuity $\exists f(x_m, y_m) \geq f$ and $f(x_m, y_m) \leq F$:

$$f(x_m, y_m) = \frac{1}{S}\int_{\mathcal{D}} f(x, y)ds.$$

Applications of the double integral

Given a region $\mathcal{D}$ in the xy-plane, we can find its area S by integrating over the region:

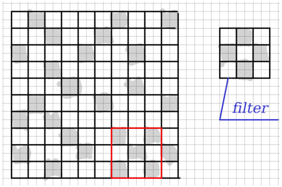
$$S = \iint_{\mathcal{D}} 1 ds$$

Double integrals define the volume of a three-dimensional region S. Divide S into thin slices parallel to the xy-plane, and integrate the area of each slice over the height of the region:

$$V = \iint_{\mathcal{D}} f(x, y) ds$$

where $f(x, y)$ gives the height of the region at each point (x, y).

Convolution integral as a filter



Suppose we have two functions $f(x,y) \in \{0,1\}$ and $g(x,y) \in \{0,1\}$. The $g(x,y)$ is defined in a frame $\mathcal{D} = [a,b] \times [c,d]$ and another one is define in the smaller frame $[\alpha, \beta] \times [\gamma, \delta]$. The

problem is to find a position (x_1, y_1) into the bigger frame $[a,b] \times [c,d]$ such that

$$\iint_{\mathcal{D}_1} f(u,v)g(x-u, y-v)dudv = \text{mes}(\mathcal{D}_1).$$

Solution of this problem give the convolution:

$$h(x,y) = \iint_{\mathcal{D}} f(u,v)g(x-u, y-v)\, du\, dv$$

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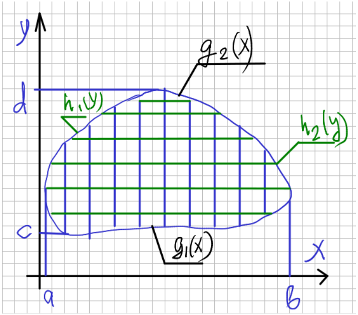
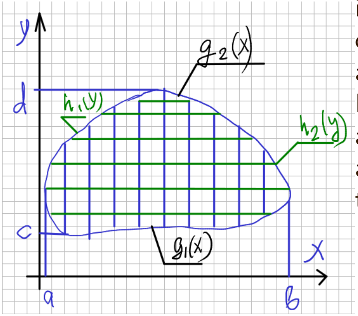
Fubini's theorem



Let $f(x,y)$ be a continuous function defined on region $\mathcal{D}$:
$\mathcal{D} = [a,b] \times [g_1(x), g_2(x)]$
or, the same,
$\mathcal{D} = [h_1(y), h_2(y)] \times [c,d]$
where
$g_{12}(x)$ and $h_{12}(y)$ continuous functions in the xy-plane.
Then the double integral of f over $\mathcal{D}$ can be expressed as an iterated integral:

$$\iint_{\mathcal{D}} f(x,y)ds = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$$

Proof the Fubini's theorem

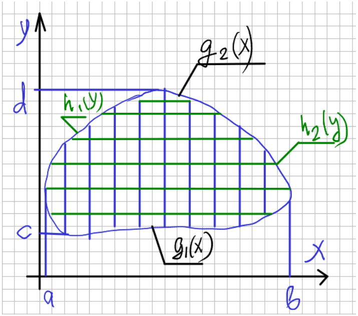


Consider the definition of a double integral. Let $P = \{(x_i, y_j)\}$ be a partition of $\mathcal{D}$, where $a = x_0 < x_1 < \cdots < x_n = b$ and $c = y_0 < y_1 < \cdots < y_m = d$. Let $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_j = y_j - y_{j-1}$ be the widths and heights of the rectangles of $\mathcal{D}$. Then the double Riemann sum of f over P is:

$$S(P, f) = \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i, y_j) \Delta x_i \Delta y_j$$

We want to show that the limit of $S(P, f)$ as the mesh size of P goes to zero is equal to the double integral of f over $\mathcal{D}$.

Proof the Fubini's theorem



Due to continuity of f on $\mathcal{D}$ one can apply the Mean Value Theorem for integrals. First, let's look at the iterated integral $\int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)dydx$. Let's integrate $f(x, y)$ with respect to y first, and then – with respect to x.

Proof the Fubini's theorem

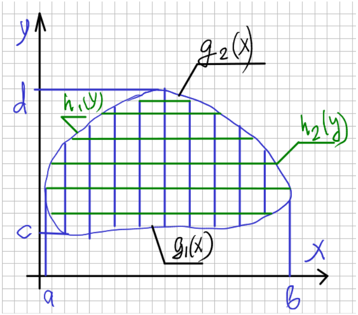
Fix x in the interval $[x_{i-1}, x_i]$. Then the Mean Value Theorem for integrals tells us that there exists a number y_i in the interval $[y_{j-1}, y_j]$ such that:

$$\int_{y_{j-1}}^{y_j} f(x,y)dy = f(x, y_i)\Delta y_j.$$

Summing over all j, one gets:

$$\int_{g_1(x)}^{g_2(y)} f(x,y)dy = \sum_{j=1}^{m(x)} f(x, y_j)\Delta y_j$$

Proof the Fubini's theorem



Substituting this
into the iterated integral, one gets:

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx =$$

$$\sum_{i=1}^{n} \sum_{j=1}^{m_i} f(x_i, y_j)\Delta x_i \Delta y_j$$

Notice that this is exactly the double Riemann sum of f over P, except that we have divided by the width of the y-interval.

Proof the Fubini's theorem

Taking the limit as the mesh size of P goes to zero, we get:

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx = \iint_{\mathcal{D}} f(x,y)ds$$

Similarly, we can show that $\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy = \iint_{\mathcal{D}} f(x,y)ds$ by integrating $f(x,y)$ with respect to x first and then with respect to y.

Repetitive integrals. Example

Consider the double integral:

$$\iint_{\mathcal{D}} (x+y)\, ds$$

where $\mathcal{D}$ is the region in the xy-plane bounded by the lines $y = x$, $y = 2$, and $x = 0$. Consider the triangle with vertices $(0,0)$, $(0,2)$, and $(2,2)$, then, we can compute the integral as follows:

$$\iint_{\mathcal{D}} (x+y)\, ds = \int_0^2 \int_x^2 (x+y)\, dy\, dx = \int_0^2 \left(xy + \frac{y^2}{2} \right) \Bigg|_{y=x}^{y=2} dx$$

$$= \int_0^2 \left(2x + 2 - \frac{3}{2}x^2 \right) dx = \left(x^2 + 2x - \frac{1}{2}x^3 \right) \Bigg|_{x=0}^{x=2} = 4.$$

Repetitive integrals. Example

Consider the double integral:

$$\iint_{\mathcal{D}} (xy)\, ds$$

where $\mathcal{D}$ is bounded by $y = x^3$ and $y = \sqrt{x}$.

$$\iint_{\mathcal{D}} xyds = \quad \int_0^1 \int_{x^3}^{\sqrt{x}} xydydx = \int_0^1 (x\tfrac{y^2}{2})\Big|_{y=x^3}^{y=\sqrt{x}} dx =$$

$$\int_0^1 (\tfrac{x^2}{2} - \tfrac{x^7}{2}) = (\tfrac{x^3}{6} - \tfrac{x^8}{16})\Big|_{x=0}^{x=1} = \frac{1}{6} - \frac{1}{16} = \frac{5}{48}.$$