Functions of several variables

O.M. Kiselev
o.kiselev@innopolis.ru

Innopolis university

February 24, 2023

Two dimensional manifold

Regions and boundaries

A limit of a function of several variables

A continuity for the functions of several variables

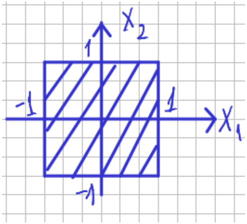
Partial derivatives

Two dimensional set

- Let as define a point of two-dimension space an element A which properties are defined by two numbers x_1 and x_2.
- This numbers might be **a mass and a temperature** of some body, or **a saturation and brightness** of a pixel and so on.
- The most important issue of the two-dimensionality is an independence of the properties.

So on such way we obtain two-dimensional set.

Chebyshev distance

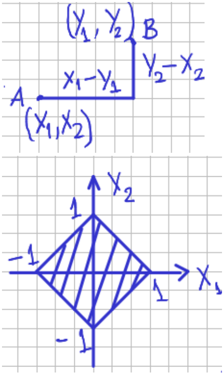


If an element is
defined by two properties of the different nature, then to define the difference two objects $A(x_1, x_2)$ and $B(y_1, y_2)$ one can consider a lot of variants.
One can define the distance $\rho(A, B)$ as **Chebyshev distance**:

$$\rho_C(A, B) = \max_{i=1,2} |x_i - y_i|.$$

The ball with radius 1 in the term of Chebyshev distance.

Manhattan distance

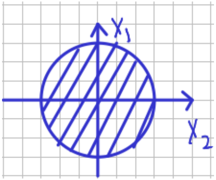


Another example of the distance might be defined as **Manhattan distance**:

$$\rho_M(A,B) = |x_1 - y_1| + |x_2 - y_2|.$$

The ball with radius 1 in terms of the Manhattan distance.

Euclidean distance

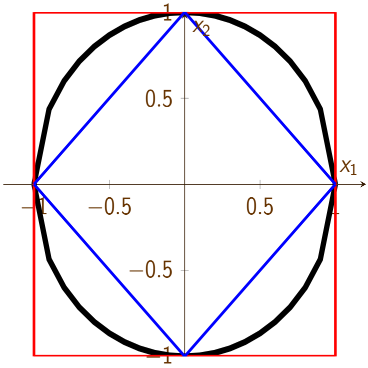


At the end
one can define as **Euclidean distance**:

$$\rho_E(A, B) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

The ball with
radius 1 in terms of Euclidean distance.

The two-dimensional unit ball



The red lines are border of of the ball of radius $R = 1$ for **Chebyshev distance.** The blue lines define the border of the ball of the radius $R = 1$ for **the Manhattan distance.** The **black line** is the border of the of the ball of the radius $R = 1$ for **the Euclidean distance.**

Properties of the distance

The defined distance has the following properties.

- $\rho(A, B) = 0 \Leftrightarrow A \equiv B$;
- $\rho(A, B) \geq 0$;
- $\rho(A, B) + \rho(B, C) \geq \rho(A, C)$.

The triangle properties for the mentioned distances

Let the coordinates of the point C be equal (z_1, z_2).

$$\max_{i\in\{1,2\}} |x_i - z_i| = \max_{i\in\{1,2\}} |x_i - z_i + y_i - y_i|$$

$$= \max_{i\in\{1,2\}} |(x_i - y_i) + (y_i - z_i)| \leq \max_{i\in\{1,2\}} |x_i - y_i| + \max_{i\in\{1,2\}} |y_i - z_i|,$$

$$\max_{i\in\{1,2\}} |x_i - y_i| + \max_{i\in\{1,2\}} |y_i - z_i| \geq \max_{i\in\{1,2\}} |x_i - z_i|,$$

$$\rho_C(C, A) \leq \rho_C(A, B) + \rho_C(B, C).$$

The triangle properties for the Euclidean distance

$$\rho_E = \sqrt{\sum_{k=1}^{N}(x_k - z_k)^2} = \sqrt{\sum_{k=1}^{N}(x_k - z_k - y_k + y_k)^2}$$
$$= \sqrt{\sum_{k=1}^{N}((x_k - y_k) + (y_k - z_k))^2}.$$

Define for simplicity: $a_k = (x_k - z_k)$, $b_k = (y_k - z_k)$. Consider:

$$\sum_{k=1}^{N}(a_k + b_k)^2 = \sum_{k=1}^{N} a_k^2 + 2\sum_{k=1}^{N}\sum_{l=1}^{N} a_k b_l + \sum_{k=1}^{N} b_k^2 \leq$$
$$\leq \left(\sqrt{\left(\sum_{k=1}^{N} a_k^2\right)^2} + \sqrt{\left(\sum_{k=1}^{N} b_k^2\right)^2}\right)^2$$

The triangle properties for the Euclidean distance

$$\rho_E(x,z) \le \sqrt{\left(\sqrt{\left(\sum_{k=1}^{N} a_k^2\right)^2} + \sqrt{\left(\sum_{k=1}^{N} b_k^2\right)^2}\right)^2} \le$$

$$\le \sqrt{\left(\sum_{k=1}^{N} a_k^2\right)^2} + \sqrt{\left(\sum_{k=1}^{N} b_k^2\right)^2}$$

$$\rho_E(x,z) \le \rho_E(x,y) + \rho_E(y,z).$$

Norm as a distance

Define a norm of the vector $X = (x_1, \dots, x_N)$.

- $||X|| = 0, \Leftrightarrow X = 0$.
- $||\lambda X|| = |\lambda| \cdot ||X||,\ \lambda \in \mathbb{R}$.
- $||X|| + ||Y|| \geq ||X + Y||$.

Theorem about equivalence of norms

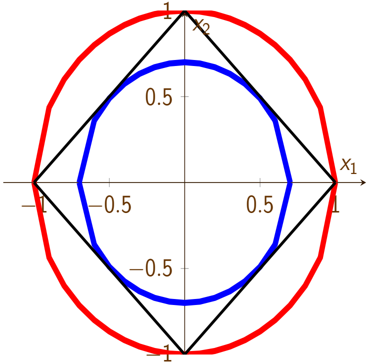


Figure: The Manhattan norm is equivalent to Euclidean one

$$\forall || \cdot ||_{1,2},\ \exists C_1, C_2 > 0 :$$

$$C_1||X||_1 \leq ||X||_2 \leq C_2||X||_1.$$

Proof of the theorem about equivalence of the norms

$$X \in \mathbb{R}^N,\ ||X||_1,\ \sup_{||X||_1=1} ||X||_2 = C_2,\ \inf_{||X||_1=1} ||X||_2 = C_1,$$

$$||X||_2 = \left|\left| ||x||_1 \frac{X}{||X||_1} \right|\right|_2 = ||X||_1 \left|\left| \frac{X}{||X||_1} \right|\right|_2,$$

$$C_1 ||X||_1 \leq ||X||_2 \leq C_2 ||X||_1.$$

$\forall ||X||_2$ equivalent $\forall ||X||_1$.

The definition of the distance

For distance between two points X and Y we will use the form:

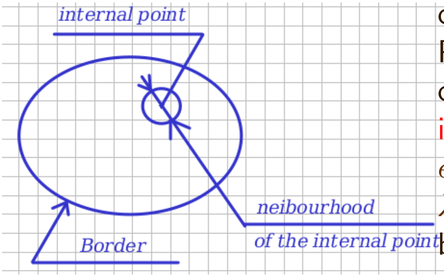
$$\rho(X, Y) \equiv ||X - Y||$$

without specification the kind of the norm. Mostly due to their equivalence of the norms we will assume the Euclidean norm:

$$||X|| = \sqrt{\sum_{k=1}^{N} x_k^2},\ X = (x_1, \ldots, x_n).$$

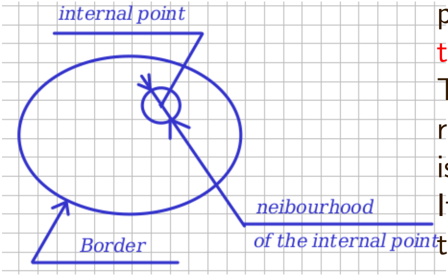
Typically the dimension $N = 2$ is enough for understanding the considered ideas.

A lot of definitions



Let's consider a region on the two dimensional set. Point X_0 of region G we define as an internal point if there exists $\epsilon > 0$, such that all points X of the region $\rho(X_0, X) < \epsilon$ belong to the region G
A region contained the internal points only is called an open region or an open set.
Points of the region which are not internal points is called border points.

A lot of definitions



All set of the border points of the region define the border of the region. The region contained the border is called closed region (set). If there exists a ball such that all points of the region contained in the ball then such region is called bounded region.

The closed bounded region (set) is called a compact region (set).

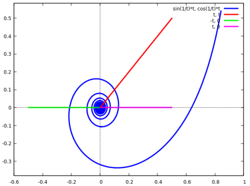
A limit of a function of two variables

If $\forall\epsilon > 0\ \exists\delta(\epsilon) : |F(Y) - A| < \epsilon\ \forall Y :\ ||Y - X|| < \delta$, then the value A is a limit of the function F in the point X:

$$\lim_{||Y-X||\to 0} f(X) = A.$$

A function which has a limit in all pints of given set is continuous function on this set.

Examples of the approaches to the given point

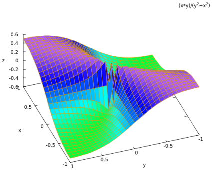


Let's consider
the origin as a point of approaching.

$$X \to 0, \Rightarrow ||X|| \to 0 :$$

1. $X = (x_1, x_2),\ ||X|| \to 0 \ \Rightarrow\ x_1 \to 0,\ x_2 \to 0;$
2. $X = (x_1, x_2),\ ||X|| \to 0 \Rightarrow x_2 \to 0,\ x_1 \to 0;$
3. $X = (x_1, x_2),\ ||X|| \to 0 \Rightarrow x_1 = kx_2 \to 0,\ x_2 \to 0;$
4. $X = (x_1, x_2),\ ||X|| \to 0 \Rightarrow x_1 = r\cos(\alpha), x_2 = r\sin(\alpha), r \to 0;$
5. $X = (x_1, x_2),\ ||X|| \to 0 \Rightarrow x_1 = r\cos(\alpha/r), x_2 = r\sin(\alpha/r), r \to 0.$

Examples of the limits



$$\lim_{X\to 0}(x_1^2+x_2^2)=0,$$

$$\lim_{X\to(2,1)}(x_1^2+x_2^2)=4+1=5;$$

$$\lim_{x_1\to 0}\lim_{x_2\to 0}\frac{x_1x_2}{x_1^2+x_2^2}=\lim_{x_1\to 0}0=0;$$

$$\lim_{x_1\to 0}\left.\frac{x_1x_2}{x_1^2+x_2^2}\right|_{x_2=kx_1}=\lim_{x_1\to 0}\frac{x_1kx_1}{x_1^2+k^2x_1^2}=\frac{k}{1+k^2},$$

$$\lim_{r\to 0}\left.\frac{x_1x_2}{x_1^2+x_2^2}\right|_{\substack{x_1=r\cos(\alpha),\\ x_2=r\sin(\alpha)}}=\lim_{r\to 0}\frac{r^2\cos(\alpha)\sin(\alpha))}{r^2}=\frac{1}{2}\sin(2\alpha).$$

Iterated limits and limit interchanging

Let's consider the iterated limits adna limit as $||x|| \to 0$:

$$\lim x_1 \to 0 \lim_{x_2 \to 0} \frac{x_1}{x_1 + x_2} = \lim_{x_1 \to 0} \frac{x_1}{x_1} = 1,$$

$$\lim_{x_2 \to 0} \lim x_1 \to 0 \frac{x_1}{x_1 + x_2} = \lim_{x_2 \to 0} 0 = 0.$$

$$\lim_{r \to 0} \frac{x_1}{x_1 + x_2} = \lim_{r \to 0} \frac{r\cos(\phi)}{r\cos(\phi) + r\sin(\phi)} = \frac{\cos(\phi)}{\cos(\phi) + \sin(\phi)}.$$

One can see both iterated limits exist but they are different and a limit as $||x|| \to 0$ does not exists.
This examples show that the changing of the iterated limits can change the answer.
The question is: When can be changed the iterated limits?

The theorem about interchanging the iterated limits

If $\exists \lim_{X\to 0} = A$ and $\exists \lim_{x_1\to 0} f(x_1, x_2) = f(0, x_2) \forall x_2 \neq 0$, then

$$\lim_{x_1\to 0}\lim_{x_2\to 0} f(x_1, x_2) = \lim_{x_2\to 0}\lim_{x_1\to 0} f(x_1, x_2) = A.$$

Proof.

$$||X|| < \delta(\epsilon) \Rightarrow |f(x_1, x_2) - A| < \epsilon \ \Rightarrow |f(x_1, 0) - A| < \epsilon, \Rightarrow$$
$$\lim_{x_1\to 0} = A, \Rightarrow \lim_{x_1\to 0}\lim_{x_2\to 0} f(x_1, x_2) = A.$$

A continue function of two variables

Define the function $f(x_1, x_2)$ as continuous in the point Y if there exist the double limit $f(X)$ as $||X - Y|| \to 0$ and

$$\lim_{||X-Y||\to 0} f(Y) = f(x_1, x_2) = A.$$

Theorem about boundedness of continuous function

A continuous function is bounded on the a bounded closed set and the function has both infimum and supremum on this set.
Proof.
Suppose there exists unbounded continuous function $f(X)$ on the bounded closed set S. Then there exists a sequence $\{X_n\}_{n=1}^{\infty}, \forall M \exists N : \; n > N \; |f(X_n)| > M$. But for close and bounded set $\forall n : \; X_n \in S \Rightarrow f(X)$ discontinuous on S which is contradiction.
Let $\sup_{X \in S} f(X) = M$ suppose $f(X) \neq M \; \forall X \in S$. Then $\phi = \frac{1}{M - f(X)}$ is continuous and unbounded on the S which contradicts to the previous statement.

Partial derivatives

Define a partial derivative of function $f(X)$:

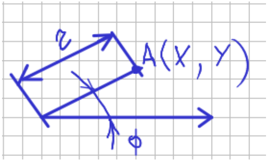
$$\frac{\partial f}{\partial x_k} \equiv \lim_{\Delta \to 0} \frac{f(x_1, \ldots, x_k + \Delta, \ldots, x_N) - f(x_1, \ldots, x_k, \ldots, x_N)}{\Delta},$$

$$\frac{\partial f}{\partial x_k} \equiv \frac{df}{dx_k} \Big|_{x_n = \text{const}, \forall n \neq k} .$$

The linear part of the function change is called differential:

$$f(X + \Delta X) - f(X) = \sum_{n=1}^{N} \frac{\partial f}{\partial x_n}\Big|_X \Delta x_n + o(||\Delta X||).$$

Polar coordinates



$$x = r\cos(\phi), \quad y = r\sin(\phi),$$
$$r = \sqrt{x^2 + y^2}, \ \phi = \arctan\left(\frac{y}{x}\right).$$
$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} = \frac{r\cos(\phi)}{r} = \cos(\phi),$$
$$\frac{\partial \phi}{\partial x} = \frac{-y}{x^2}\frac{1}{1 + \left(\frac{y}{x}\right)^2} = -\frac{r\sin(\phi)}{x^2 + y^2} = -\frac{\sin(\phi)}{r};$$
$$\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}} = \frac{r\sin(\phi)}{r} = \sin(\phi),$$
$$\frac{\partial \phi}{\partial y} = \frac{1}{x}\frac{1}{1 + \left(\frac{y}{x}\right)^2} = \frac{r\cos(\phi)}{x^2 + y^2} = \frac{\cos(\phi)}{r};$$

Polar coordinates

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial x} = \frac{\partial f}{\partial r}\cos(\phi) - \frac{\partial f}{\partial \phi}\frac{\sin(\phi)}{r};$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial y} = \frac{\partial f}{\partial r}\sin(\phi) + \frac{\partial f}{\partial \phi}\frac{\cos(\phi)}{r}.$$

Polar coordinates

$$
\begin{aligned}
dx &= \cos(\phi)dr - r\sin(\phi)d\phi, \;\; dy = \sin(\phi)dr + r\cos(\phi)d\phi; \\
df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \\
& \left(\frac{\partial f}{\partial r}\cos(\phi) - \frac{\partial f}{\partial \phi}\frac{\sin(\phi)}{r}\right)(\cos(\phi)dr - r\sin(\phi)d\phi) + \\
& \left(\frac{\partial f}{\partial r}\sin(\phi) + \frac{\partial f}{\partial \phi}\frac{\cos(\phi)}{r}\right)(\sin(\phi)dr + r\cos(\phi)d\phi) = \\
&= \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \phi}d\phi.
\end{aligned}
$$

Summary

Two dimensional manifold

Regions and boundaries

A limit of a function of several variables

A continuity for the functions of several variables

Partial derivatives