Lecture 4. Fourier series

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Periodic functions

A piecewise continuous function $f(x)$ such that $f(x+T) = f(x) \forall x \in \mathbb{R}$ for certain constant $T > 0$ is called periodic function.
The value $T > 0$ is called period.
Let T is a period of $f(t)$, then quantities $\tilde{T} = nT, \forall n \in \mathbb{Z}$ are periods of $f(x)$.
The reciprocal quantity is called frequency

$$\omega = \frac{1}{T}.$$

Periodic functions

Theorem

Let T be the smallest of the periods of a piecewise continuous function $f(t)$, then all periods of the function are nT, $\forall n \in \mathbb{Z}$ or the function is a constant.

Proof. Suppose the function f has two different periods. Define the smallest one as T_1 and another one as and $T_2 \neq nT_1, \;\forall n \in \mathbb{N}$.

$$f(t - T_1)) = f(t),\; f(t - T_1 + T_2) = f(t + (T_2 - T_1)) = f(t), \Rightarrow$$
$$T_2 - T_1 \text{ is a period}, \Rightarrow \exists N > 0 : T_1 > T_2 - NT_1 > 0,$$
$$T = T_2 - NT_1,\; f(t + T) = f(t).$$

We obtain a contradiction.

Properties of periodic functions

Let two functions $f(t)$ and $h(t)$ be periodic. Their periods are T_1 and T_2 correspondingly. The sum of the functions

$$g(t) = f(t) + h(t),$$

is a periodic functions of period T if $\exists n, m \in \mathbb{N}: \; T_1 n = T_2 m$ and $T = T_1 n = T_2 m$.

Examples:

$$\begin{aligned} g(t) &= \sin(3t) + \cos(5t) \;\; T_1 = \frac{2\pi}{3}, \;\; T_2 = \frac{2\pi}{5}, \\ & \quad 3T_1 = 5T_2 = 2\pi, \;\; T = 2\pi. \\ y(t) &= \sin(3t) + \cos(\sqrt{2}t) \;\; T_1 = \frac{2\pi}{3}, \; T_2 = \frac{2\pi}{\sqrt{2}}, \; \frac{T_1}{T_2} \notin \mathbb{Q} \;\Rightarrow \\ y(t) & \quad \text{has no period.} \end{aligned}$$

Periodic functions of period 2π

Let $f(x)$ be periodic with a period T.
If one changes the variable $y = 2\pi x/(T)$, then one will get $2\pi(x + T)/T = 2\pi x/T + 2\pi$. Therefore the changing of the variable $y = 2\pi x/T$ define new 2π- periodic function:

$$f(y + 2\pi) = f(y).$$

Below we consider the functions with period 2π.

Definition of Fourier series

The series

$$S(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx).$$

It is easy to see at specific points the the series turn into the following:

$$S(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n, \quad x = 2\pi k,\ k \in \mathbb{Z};$$

$$S(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} b_n, \quad x = \frac{\pi}{2} + 2\pi k,\ k \in \mathbb{Z}.$$

In this case the series are absolutely convergent uniformly on the interval $x \in [0, 2\pi)$ if both series contained a_n and b_n are absolutely convergent.

Fourier approximation of cosine by series of sines.

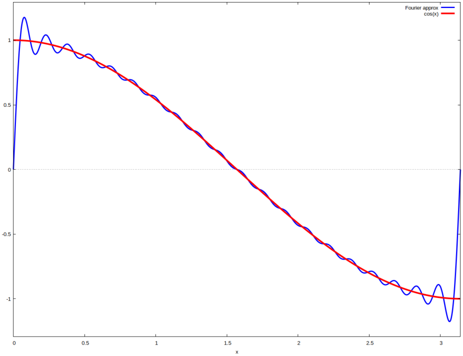

Figure: Fourier approximation of the cosine by series of sine on a half of period: $\cos(x)$ and $\frac{8}{\pi}\sum_{n=1}^{20}\frac{n}{4n^2-1}\sin(nx)$.

Integration of the Fourier series

$$\int_{x0}^{x} S(t)dt = \int_{x0}^{x} \left(\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt) \right) dt =$$

$$= \int_{x0}^{x} \left(\frac{1}{2}a_0 + \sum_{n=1}^{N} a_n \cos(nt) + b_n \sin(nt) \right) dt +$$

$$+ \int_{x0}^{x} \left(\sum_{n=N}^{\infty} a_n \cos(nt) + b_n \sin(nt) \right) dt.$$

$$\forall \epsilon > 0,\ \exists N : \left| \sum_{n=N}^{\infty} a_n \cos(nt) + b_n \sin(nt) \right| < \epsilon \Rightarrow$$

$$\int_{x0}^{x} S(t)dt = \frac{1}{2}(x - x_0)a_0 + \sum_{n=1}^{N} \int_{x0}^{x} (a_n \cos(nt) + b_n \sin(nt))\, dt + \mathcal{O}(\epsilon).$$

Integrating of the Fourier series

$$
\begin{aligned}
\int_{x0}^{x} S(t)dt = (x - x_0)a_0 \; &+ \; \sum_{n=1}^{\infty}\Big(\frac{a_n}{n}(\sin(nx) - \sin(nx_0)) - \\
&- \; \frac{b_n}{n}(\cos(nx) - \cos(nx_0))\Big) = \\
= \frac{1}{2}(x - x_0)a_0 \; &+ \; \sum_{n=1}^{\infty}\left(\frac{a_n}{n}\sin(nx) - \frac{b_n}{n}\cos(nx)\right) \\
&- \; \sum_{n=1}^{\infty}\left(\frac{a_n}{n}\sin(nx_0) - \frac{b_n}{n}\cos(nx_0)\right)
\end{aligned}
$$

Theorem about integration of the Fourier series

If Fourier series absolutely convergent uniformly over the period, then the integral of the series is equal to the Fourier series integrated term by term.

$$\int_{x0}^{x} \left(\frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt) \right) dt$$

$$= \frac{1}{2}(x - x_0) a_0 + \sum_{n=1}^{\infty} \left(\frac{a_n}{n} \sin(nx) - \frac{b_n}{n} \cos(nx) \right) -$$

$$- \sum_{n=1}^{\infty} \left(\frac{a_n}{n} \sin(nx_0) - \frac{b_n}{n} \cos(nx_0) \right)$$

Differentiating of the Fourier series

If the coefficients of the Fourier series a_n and b_n are such that both series

$$\sum_{n=1}^{\infty} n|a_n|, \ \sum_{n=1}^{\infty} n|b_n|$$

are convergent, then the Fourier series the derivative of the Fourier sum is equal to differentiated term by term series.

Proof.

$$\begin{aligned} S(x) &= \int_{x_0}^{x} \sum_{n=1}^{\infty} \left(-na_n \sin(nt) + nb_n \cos(nt)\right) dt \\ &= \sum_{n=1}^{\infty} \left(a_n \cos(nt) + b_n \sin(nx)\right) \Rightarrow \\ \frac{d}{dx} S(x) &= \frac{d}{dx} \sum_{n=1}^{\infty} \left(-na_n \sin(nx) + nb_n \cos(nx)\right). \end{aligned}$$

Fourier approximation

Let assume $f(x)$ be a represented as a Fourier series:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx).$$

Then the coefficients of the Fourier series are:

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)dx, \ \ n \in \mathbb{N};$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)dx, \ \ n \in \mathbb{N}.$$

Formulae for the Fourier coefficients

$$\int_0^{2\pi} \cos(mx)\cos(nx)dx = \int_0^{2\pi} \left(\frac{1}{2}\cos\left(\frac{x(m-n)}{2}\right) + \right.$$

$$\left. + \frac{1}{2}\cos\left(\frac{x(m+n)}{2}\right)\right) dx = \begin{cases} \pi, \ n = m; \\ 0, \ n \neq m. \end{cases}$$

Formulae for the Fourier coefficients

$$\int_0^{2\pi} \sin(mx)\sin(nx)dx = \int_0^{2\pi} \left(\frac{1}{2}\cos\left(\frac{x(m-n)}{2}\right) - \right.$$

$$\left. - \frac{1}{2}\cos\left(\frac{x(m+n)}{2}\right)\right) dx = \begin{cases} \pi, \ n = m; \\ 0, \ n \neq m. \end{cases}$$

Formulae for the Fourier coefficients

$$\int_0^{2\pi} \cos(mx)\sin(nx)dx = \int_0^{2\pi} \left(\frac{1}{2}\sin\left(\frac{x(m+n)}{2}\right) - \right.$$
$$\left. - \frac{1}{2}\sin\left(\frac{x(m-n)}{2}\right)\right) dx = 0.$$

Fourier approximation of the step-function

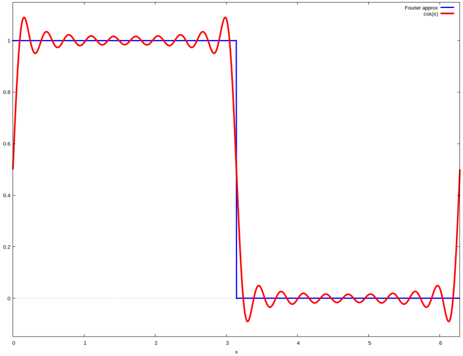

Figure: Fourier approximation for the curve $\frac{1}{2}(1 + \text{sign}(\pi - x))$ by $\frac{1}{2} + \frac{1}{\pi} \sum_{n=1}^{10} \frac{\sin((2n-1)x)}{2n-1}$. The Gibbs phenomenon is the oscillatory behavior of the piecewise differentiable periodic function around jump discontinuity.

Behavior of the Fourier coefficients

If a periodic function has derivatives of k-th order, then

$$a_n = \mathcal{O}(n^{-k}),\ b_n = \mathcal{O}(n^{-k}),$$

$$\begin{aligned}
a_n &= \frac{1}{2\pi}\int_0^{2\pi} f(x)\cos(nx)dx = \\
&= \frac{f(x)}{2\pi n}\sin(nx)\Big|_{x=0}^{x=2\pi} - \frac{1}{2\pi n}\int_0^{2\pi} f'(x)\sin(nx)dx \\
&= -\frac{f''(x)}{2\pi n^2}\cos(nx)\Big|_{x=0}^{x=2\pi} + \frac{1}{2\pi n^2}\int_0^{2\pi} f''(x)\cos(nx)dx = \\
&= \ldots .
\end{aligned}$$

Example. Fourier approximation for smooth function

$$
\begin{aligned}
f(x) &= x(\pi - x)(2\pi - x), \\
a_0 &= \frac{1}{2\pi}\int_0^{2\pi} x(\pi - x)(2\pi - x)dx = 0; \\
a_n &= \frac{1}{\pi}\int_0^{2\pi} x(\pi - x)(2\pi - x)\cos(nx)dx = \frac{12}{n^3}; \\
f(x) &= 12\sum_{k=0}^{\infty} \frac{\sin(nx)}{n^3}.
\end{aligned}
$$

Example. Fourier approximation for smooth function

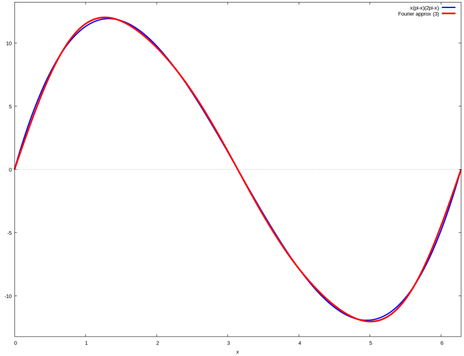

Figure: $f(x) = x(\pi - x)(2\pi - x)$ and $s(x) = \sum_{n=1}^{3} \frac{\sin(nx)}{n^3}$.

Example. Fourier approximation for smooth function

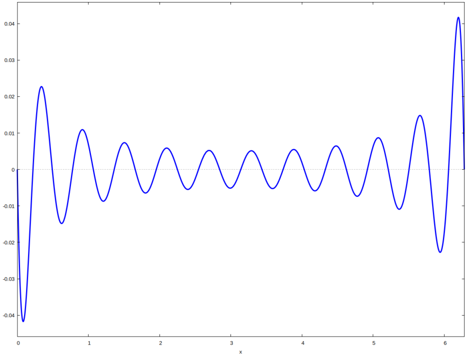

Figure: $f(x) = x(\pi - x)(2\pi - x) - S(x) = \sum_{n=1}^{10} \frac{sin(nx)}{n^3}$.

Euler's formula

$$
\begin{aligned}
i &= \sqrt{-1}, \quad i^2 = -1. \\
e^{ix} &= \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{n=0}^{\infty} \frac{i^n\, x^n}{n!}, \\
& \quad i^3 = -i,\ i^4 = 1,\ i^5 = i, i^6 = -1, \ldots \\
& \quad i^{2n} = (-1)^n, \quad i^{2n+1} = (-1)^n i, \\
e^{ix} &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}. \\
& \quad e^{ix} = \cos(x) + i \sin(x). \\
& \quad e^{inx} = \cos(nx) + i \sin(nx).
\end{aligned}
$$

Fourier series in complex form

Using the Euler formula one can write:

$$\cos(nx) = \frac{e^{inx} + e^{-inx}}{2}, \ \sin(nx) = \frac{e^{inx} - e^{-inx}}{2i}.$$

Then the Fourier series one can rewrite:

$$\begin{aligned} S(x) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) \\ &= \sum_{n=1}^{\infty} a_n \frac{e^{inx} + e^{-inx}}{2} + b_n \frac{e^{inx} - e^{-inx}}{2i} = \\ &= \sum_{n=1}^{\infty} \left(\frac{a_n}{2} + \frac{b_n}{2i}\right) e^{inx} + \left(\frac{a_n}{2} - \frac{b_n}{2i}\right) e^{-inx} = \end{aligned}$$

$$\frac{1}{2i} \cdot \frac{i}{i} = \frac{i}{-2} = \frac{-1}{2} \Rightarrow$$

Fourier series in complex form

$$
\begin{aligned}
S(x) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(\frac{a_n}{2} + \frac{-ib_n}{2}\right)e^{inx} + + \left(\frac{a_n}{2} + \frac{ib_n}{2i}\right)e^{-inx} = \\
&= \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(\frac{a_n}{2} + \frac{-ib_n}{2}\right)e^{inx} + \sum_{n=1}^{\infty}\left(\frac{a_n}{2} + \frac{ib_n}{2i}\right)e^{-inx} \\
&= \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(\frac{a_n}{2} + \frac{-ib_n}{2}\right)e^{inx} + \sum_{n=-\infty}^{-1}\left(\frac{a_{-n}}{2} + \frac{ib_{-n}}{2i}\right)e^{inx}.
\end{aligned}
$$

Define $c_0 = a_0,\ c_n = (a_n - ib_n)/2,\ c_{-n} = (a_n + ib_n)/2,\ n \in \mathbb{N}$, then one obtains:

$$S(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

Fourier coefficients in complex form

$$\frac{1}{2}(a_n - ib_n) = \frac{1}{2\pi}\int_0^{2\pi} f(x)(cos(nx) - i\sin(nx))dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx,$$

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx,$$

if $\sum_{n=-\infty}^{\infty} |c_n|$ converges absolutely, then:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

Orthogonality of the complex exponents

Assume $n, m \in \mathbb{N}$:

$$\frac{1}{2\pi}\int_0^{2\pi} e^{inx}e^{-imx}dx = \int_0^{2\pi} e^{i(n-m)x}dx =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cos((n-m)x) + i\sin((n-m)x)dx =$$

$$= \begin{cases} 1, \ n = m \\ 0, \ n \neq m. \end{cases}.$$

Fourier series for T-periodic functions

If $f(x) = f(x+T)$ and T is the smallest period of the function $f(x)$, then the Fourier series for this function can be constructed by following formulae:

$$a_n = \frac{2}{T}\int_0^T f(x)\cos\left(\frac{2\pi}{T}xn\right)dx,$$

$$b_n = \frac{2}{T}\int_0^T f(x)\sin\left(\frac{2\pi}{T}xn\right)dx,$$

$$S(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos\left(\frac{2\pi}{T}xn\right) + b_n\sin\left(\frac{2\pi}{T}xn\right).$$

Fourier series for T-periodic functions in complex form

$$c_n = \frac{1}{T}\int_0^T f(x) e^{-i\frac{2\pi}{T}nx}\,dx,$$

$$S(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi}{T}xn}.$$

Multiplication and convolution

$$S(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \; Q(x) = \sum_{n=-\infty}^{\infty} q_n e^{inx},$$

$$S(x)Q(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \sum_{k=-\infty}^{\infty} q_k e^{ikx} = \sum_{m=-\infty}^{\infty} p_m e^{inx},$$

$$p_m = \sum_{k+n=m} c_n q_k, \quad p_m = \sum_{n=-\infty}^{\infty} c_n q_{m-n}.$$

Summary

Periodic functions

Definition of Fourier series

Fourier approximation

Complex form of the Fourier series