Lecture 3. Power series

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Power series

Differentiating and integration

Multiplication and convolutions

Solutions of differential equations

Definitions

The power series has a form:

$$S(x) = \sum_{n=0}^{\infty} a_n x^n \text{ or } S(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

Changing of the variable $x - x_0 \to \xi$ allows one to rewrite the right-hand form of the power series to the left-hand one. Therefore we will study the left-hand side formula for the power series.
Here x is considered as independent variable and properties f the series depend on the value of x.

$$S(x) = \sum_{n=0}^{\infty} x^n,$$

$$P(x) = \sum_{n=0}^{\infty} \frac{n^2}{n^3 + 1} x^n.$$

Counterexamples

$$f(x) = e^{-1/x^2},\ x \neq 0,\ f(0) = 0.$$

$$f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}, \quad a_n = \frac{d^n}{dx^n}(e^{-1/x^2})\big|_{x=0} = 0.$$

$$Q(x) = \sum_{n=0}^{\infty} \frac{x^n \sin(nx)}{n+1},$$ is a functional but it is not a power series.

The tests of convergence

A convergence of power series can be established by the same tests like one for the numerical series.

The ratio test:

The series converges for the values of x:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| |x| < 1.$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \rho \Rightarrow |x| < \frac{1}{\rho}.$$

The series absolute converges $\forall x : |x| < \frac{1}{\rho}$ and diverges $\forall x : |x| > \frac{1}{\rho}$.

Root test

$$\lim_{n\to\infty} \sqrt[n]{|a_n x^n|} = \lim_{n\to\infty} \sqrt[n]{|a_n|}\,|x| < 1.$$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \rho \Rightarrow |x| < \frac{1}{\rho}.$$

The series absolute converges $\forall x : |x| < \frac{1}{\rho}$ and diverges $\forall x : |x| > \frac{1}{\rho}$.

Convergence of the power series

Theorem about the radius of convergence

If the series power series

$$S(x) = \sum_{n=0}^{\infty} a_n x^n$$

converges for $x = x_0$, then this series absolute converges in the interval $|x| < |x_0|$.

The largest value R of $|x|$ for the convergence of the power series is called the **radius of convergence**.

The radius of convergence. Examples

$$\sum_{n=1}^{\infty} \frac{x^n}{n}, \; R = 1;$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n!}, \; \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{(n+1)} = 0 \Rightarrow R = \infty,$$

$$\sum_{n=1}^{\infty} x^n n!, \; \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} (n+1) = \infty \Rightarrow R = 0.$$

Proof the theorem about interval of convergence

If the series converges at $x = x_0$, then

$$\lim_{n\to\infty} a_n x_0^n \to 0 \;\Rightarrow \exists N, \forall n > N: \; |a_n| < \frac{1}{|x_0|^n}.$$

Due to the comparison theorem: $\displaystyle\sum_{n=N}^{\infty} a_n x^n < \sum_{n=N}^{\infty} \left|\frac{x}{x_0}\right|^n.$

$$\forall x: \; \left|\frac{x}{x_0}\right| < 1, \text{ hence the last series converges.}$$

Then the series $S(x)$ absolutely converges $\forall x: \; |x| < |x_0|$.

The Cauchy -Hadamard theorem

The radius of convergence and the root test

If R is the radius of convergence, then

$$\frac{1}{R} = \rho = \overline{\lim_{n=0}} \sqrt[n]{|a_n|}.$$

Proof. We consider only case $0 < \rho < \infty$. Then
$\forall \epsilon > 0\ \exists N, \forall n > N\ |a_n||x|^n \leq (\rho + \epsilon)^n |x|^n, \forall |x| < 1/(\rho + \epsilon)$.
Hence:

$$\sum_{n=0}^{\infty} |a_n||x|^n \leq \sum_{n=0}^{\infty} q^n,\ \ q = (\rho + \epsilon)|x| < 1.$$

The Cauchy -Hadamard theorem. Proof.

Vice versa: $\forall N \, \exists \epsilon > 0$, $n, n_l > N : \{a_{n_l}\} \subset \{a_n\} : \; |a_{n_l}| > (\rho - \epsilon)|x^{n_l}| > 1, |x| > 1/(\rho + \epsilon)$, then the series diverges for $|x| > 1/\rho$.

Derivative of the power series

Theorem about differentiating

The radius of convergence R does not changed for term by term differentiated power series.

Proof.

$$\sum_{n=0}^{\infty} \frac{d}{dx}(a_n x^n) = \sum_{n=0}^{\infty} n a_n x^{n-1}.$$

$$\lim_{n\to\infty} \sqrt[n]{n|a_n|} = \lim_{n\to\infty} \sqrt[n]{n} \lim_{n\to\infty} \sqrt[n]{|a_n|},$$

$$\lim_{n\to\infty} \log(\sqrt[n]{n}) = \lim_{n\to\infty} \frac{\log(n)}{n} = 0 \Rightarrow \lim_{n\to\infty} \sqrt[n]{n} = 1,$$

$$\lim_{n\to\infty} \sqrt[n]{n|a_n|} = \frac{1}{R}.$$

Theorem about differentiating of a function in the form of the series

Let $S(x)$ be presented in the form of power series:

$$S(x) = \sum_{n=0}^{\infty} a_n x^n.$$

and R is the radius of convergence of the series. Then

$$S'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}, \forall x,\ |x| < R.$$

Theorem about differentiating of a function in the form of series. **Proof.**

$$\begin{aligned}
S'(x) = \lim_{\Delta\to 0}\frac{1}{\Delta}\left(\sum_{n=0}^{\infty} a_n(x+\Delta)^n - \sum_{n=0}^{\infty} a_n x^n\right) = \\
\lim_{\Delta\to 0}\frac{1}{\Delta}\left(\sum_{n=0}^{\infty} a_n((x+\Delta)^n - x^n)\right) = \\
\lim_{\Delta\to 0}\frac{1}{\Delta}\left(\Delta\sum_{n=0}^{\infty}\left(a_n n x^{n-1} + \Delta a_n \sum_{k=2}^{n}\frac{n!}{k!(n-k)!}\Delta^{k-2}x^{n-(k-2)}\right)\right) = \\
\sum_{n=0}^{\infty} a_n n x^{n-1} + \lim_{\Delta\to 0}\Delta\sum_{n=2}^{\infty}\left(\sum_{k=2}^{n}\frac{n!}{k!(n-k)!}\Delta^{k-2}x^{n-(k-2)}\right)a_n.
\end{aligned}$$

Theorem about differentiating of a function in the form of series

Proof.

$$n(n+1)|\Delta + x|^{n-2} = n(n+1)\sum_{k=0}^{n-2}\frac{(n-2)!}{k!(n-2-k)!}|\Delta|^k|x|^{n-(k-2)},$$

$$\sum_{k=0}^{n-2}\frac{n!}{k!(n-2-k)!}|\Delta|^k|x|^{n-k} > \sum_{k=2}^{n}\frac{n!}{k!(n-k)!}|\Delta|^{k-2}|x|^{n-(k-2)} \Rightarrow$$

$$n(n+1)|\Delta + x|^{n-2} > \sum_{k=2}^{n}\frac{n!}{k!(n-k)!}|\Delta|^{k-2}|x|^{n-(k-2)},$$

$$\lim_{\Delta\to 0}\Delta\sum_{n=2}^{\infty}\left(\sum_{k=2}^{n}\frac{n!}{k!(n-k)!}\Delta^{k-2}x^{n-(k-2)}\right)|a_n| <$$

$$< \lim_{\Delta\to 0}\Delta\sum_{n=2}^{\infty}n(n+1)|\Delta + x|^{n-2}|a_n|$$

Theorem about differentiating of a function in the form of series

$$\forall x : |x| < R,\ \exists \Delta : \lim_{n \to \infty} \frac{(n+1)(n+2)|a_{n+1}|}{n(n+1)|a_n|} |\Delta + x| = \frac{|\Delta + x|}{R} < 1.$$

The series converges and the limit equals 0. Then

$$S'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Differentiating of power series. Examples

$$\frac{d}{dx}\sum_{n=1}^{\infty}\frac{x^n}{n}=\sum_{n=1}^{\infty}x^{n-1}=\sum_{n=0}^{\infty}x^n=\frac{1}{1-x}.$$

$$\frac{d}{dx}\left(x\sum_{n=0}^{\infty}\frac{x^n}{(n+1)^2}\right)=\sum_{n=0}^{\infty}\frac{x^n}{n+1}=$$

$$\frac{1}{x}\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}=\frac{1}{x}\sum_{n=1}^{\infty}\frac{x^n}{n}=-\frac{\log(1-x)}{x}.$$

Counterexample:

$$\frac{d}{dx}\sum_{n=1}^{\infty}\frac{\sin(nx)}{n^2}\neq\sum_{n=1}^{\infty}\frac{d}{dx}\left(\frac{\sin(nx)}{n^2}\right)=\sum_{n=1}^{\infty}\frac{\cos(nx)}{n}.$$

The right-hand side series diverges at $x=2\pi$.

Integration of the power series

Theorem about Integrating of the power series

The radius of convergence R does not changed for term by term integrated power series.

Proof. Consider term by term antiderivative of the series:

$$\sum_{n=0}^{\infty} \int a_n x^n dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}.$$

$$\lim_{n\to\infty} \sqrt[n]{\frac{|a_n|}{(n+1)}} = \frac{\lim_{n\to\infty} \sqrt[n]{|a_n|}}{\lim_{n\to\infty} \sqrt[n]{n+1}},$$

$$\lim_{n\to\infty} \log(\sqrt[n]{n+1}) = \lim_{n\to\infty} \frac{\log(n+1)}{n} = 0 \Rightarrow \lim_{n\to\infty} \sqrt[n]{n+1} = 1,$$

$$\lim_{n\to\infty} \sqrt[n]{\frac{|a_n|}{(n+1)}} = \frac{1}{R}.$$

Corollary. Definite integrals over the interval of convergence

$$S(x) = \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1} \Rightarrow S'(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Let R be a radius of convergence for the series

$$S'(x) = \sum_{n=0}^{\infty} a_n x^n.$$

If $-R < \alpha < \beta < R$ then:

$$\int_{\alpha}^{\beta} \sum_{n=0}^{\infty} a_n x^n \, dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (\beta^n - \alpha^n).$$

Integration of series. Examples

$$\int_0^x \sum_{n=0}^{\infty} t^n dt = \sum_{n=0}^{\infty} \int_0^x t^n dt \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1},$$

$$\int_0^x e^{-t^2} dt = \int_0^x \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{n!} dt =$$

$$\sum_{n=0}^{\infty} (-1)^n \int_0^x \frac{t^{2n}}{n!} dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1) \cdot n!}$$

Multiplication of power series

Let R is the radius of convergence for both power series:

$$A(x) = \sum_{n=0}^{\infty} a_n x^n, \;\; B(x) = \sum_{n=0}^{\infty} b_n x^n.$$

Then:

$$\sum_{n=0}^{\infty} a_n x^n \cdot \sum_{m=0}^{\infty} b_m x^m = \lim_{N\to\infty} \lim_{M\to\infty} \sum_{n=0}^{N} \sum_{m=0}^{M} a_n x^n b_m x^m =$$

$$\lim_{N\to\infty} \lim_{M\to\infty} \sum_{n=0}^{N} \sum_{m=0}^{M} a_n b_m x^{n+m} = \lim_{N\to\infty} \sum_{n=0}^{N} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n =$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n.$$

A convolution

The sum

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

is called the **convolution**.
An example. $a_n = (-1)^n$, $b_n = (1/n!)$:

$$c_0 = 1,\; c_1 = 1 - 1 = 0,\; c_2 = \frac{1}{2} - 1 + 1 = \frac{1}{2},$$

$$c_3 = \frac{1}{6} - 1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{1} - 1 = \frac{1}{3},$$

$$c_n = \sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!}.$$

Differential equations

Definition

An equation which contains derivatives of unknown function is called differential equation.

The differential equations define behavior of a lot things of different nature like sociology, biology, physics.

$$\frac{du}{dx} = -u,$$

$$\frac{d^2u}{dx^2} = -u,$$

$$\frac{d^2u}{dx^2} = -\sin(u) - \mu u.$$

Power series for the differential equations

Let's find the solution for the equation:

$$\frac{du}{dx} = ku,\ \ u = \sum_{n=0}^{\infty} u_n x^n, \Rightarrow \sum_{n=1}^{\infty} n u_n x^{n-1} = k \sum_{n=0}^{\infty} u_n x^n,$$

$$n u_n = k u_{n-1},\ u_{n+1} = \frac{k}{n} u_n, u_0 = \text{const},\ \ u_n = \frac{k^n}{n!} u_0,\ n > 0.$$

$$u(x) = \sum_{n=0}^{\infty} \frac{k^n}{n!} x^n u_0 = u_0 \sum_{n=0}^{\infty} \frac{(kx)^n}{n!} = u_0 e^{kx}.$$

Example. The Weber equation

$$\frac{d^2u}{dx^2} = (\frac{x^2}{4} + a)u, \;\; u = \sum_{n=0}^{\infty} u_n x^n,$$

$$\sum_{n=2}^{\infty} n(n-1)u_n x^{n-2} = \sum_{n=0}^{\infty} u_n(\frac{x^2}{4} + a)x^n,$$

$$2u_2 = au_0, \;\; 2 \cdot 3 \cdot u_3 = au_1,$$

$$3 \cdot 4 \cdot u_4 = au_2 + \frac{1}{4}u_0, \quad 4 \cdot 5 \cdot u_5 = au_3 + \frac{1}{4}u_1,$$

Example. The Weber equation

For general case one obtains:

$$\sum_{n=2}^{\infty} n(n-1)u_n x^{n-2} = \sum_{n=0}^{\infty} u_n\left(\frac{x^2}{4} + a\right)x^n,$$

$$u_{n+4} = \frac{u_n + au_{n+2}}{4(n+4)(n+3)},$$

$$\lim_{n\to\infty} \frac{u_{n+4}}{u_n + au_{n+2}} = \lim_{n\to\infty} \frac{1}{4(n+4)(n+3)} = 0.$$

Therefore the solution can be represented by two different series with odd and even powers of x and which have infinite radius of convergence.

The reason of the success for solving DE

The set $\{x^n\}_{n=0}^{\infty}$ one must consider as a basis set of linear independent vectors in some infinite dimensional space of function.
However, the power series does not define a function in general sense. One can always add function with zero coefficients:

$$h(x) = \sum_{n=0}^{\infty} a_n x^n,$$

$$h(x) + f(x) = \sum_{n=0}^{\infty} a_n x^n, \text{ where}$$

$$f(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!}, \quad f_n = \frac{d^n}{dx^n}(e^{-1/x^2})|_{x=0} = 0.$$

Summary

Power series

Differentiating and integration

Multiplication and convolutions

Solutions of differential equations