Absolute and conditional convergence

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Riemann series theorem

Consider the series

$$S = \sum_{n=0}^{\infty} (-1)^n u_n,$$

where $u_n > u_{n+1} > 0$, $u_n \to 0$ as $n \to \infty$ and both series

$$s_+ = \sum_{n=0}^{\infty} u_{2n}, \quad s_- = \sum_{n=0}^{\infty} u_{2n+1}$$

diverge.
Then one can rearrangement of the series such way, that the sum might be any number.

An example

$$S = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n},\ s_+ = \sum_{k=0}^{\infty} \frac{1}{2k},\ s_- = \sum_{k=0}^{\infty} \frac{1}{2k+1}$$

The integral test claims both s_+ and s_- are divergent series. Let A be equals 1.5 and construct the sum of alternating series:

$$\begin{aligned} 1.5 &= 1 + \frac{1}{3} + \frac{1}{5} \sim 1.53(3) - \frac{1}{2}(\sim 1.0(3)) + \\ &+ \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15}(\sim 1.522) - \frac{1}{4}(\sim 1.27) + \\ &+ \frac{1}{17} + \frac{1}{19} + \frac{1}{21} + \frac{1}{23} + \frac{1}{25}(\sim 1.514) - \frac{1}{6}(\sim 1.34) + \dots . \end{aligned}$$

Riemann series theorem. Proof

Take a positive number A

$$s_1 = u_0 + u_2 + \cdots + u_{2k_1} > A,$$

Here k_1 such that $u_0 + u_2 + \cdots + u_{2k_1-2} < A$.

$$s_2 = u_0 + u_2 + \cdots + u_{2k_1} - u_1 - u_3 - \cdots - u_{2l_1+1} < A,$$

Here l_1 such that
$u_0 + u_2 + \cdots + u_{2k_1-2} - u_1 - u_3 - \cdots - u_{2l_1-1} > A$.

Riemann series theorem. Proof

Further one get:

$$s_3 = \sum_{n=0}^{k_1} u_{2n} - \sum_{n=0}^{l_1} u_{2n+1} + \sum_{n=k_1+1}^{k_2} u_{2n} > A,$$

$$s_4 = \sum_{n=0}^{k_1} u_{2n} - \sum_{n=0}^{l_1} u_{2n+1} + \sum_{n=k_1+1}^{k_2} u_{2n} - \sum_{n=l_1+1}^{l_2} u_{2n+1} < A,$$

$$\dots .$$

Due to the property $u_n \to 0$ as $n \to \infty$ one can obtain

$$\lim_{n\to\infty} s_n = A.$$

Absolute convergence series

Let the series be absolute converging:

$$S = \sum_{n=0}^{\infty} a_n,$$

and let's consider the series with the same, but permuted terms:

$$\tilde{S} = \sum_{n=0}^{\infty} \tilde{a}_n,$$

Theorem

$$S = \tilde{S}.$$

Theorem about absolute convergence. A proof

Denote

$$\sigma = \sum_{n=0}^{\infty} |a_n|, \quad \tilde{\sigma} = \sum_{n=0}^{\infty} |\tilde{a}_n|.$$

$\forall m \, \exists n \geq m$, $\{\tilde{a}_k\}_{k=0}^{m} \subset \{a_k\}_{k=0}^{n}$, then: $\tilde{\sigma}_m \leq \sigma_n, \Rightarrow \{\tilde{\sigma}_m\}_{m=0}^{\infty}$ is the increasing bounded sequence, hence the sequence has a limit $\tilde{\sigma} \leq \sigma$.
The same reasoning gives the inequality $\sigma \leq \tilde{\sigma}$.

$$\sum_{n=0}^{\infty} |a_n| = \sum_{n=0}^{\infty} |\tilde{a}_n|$$

Theorem about absolute convergence. A proof

Let's consider a sequence of partial series S_n due to the absolute convergence one gets:

$$\forall \epsilon > 0 \, \exists N : \forall n > N \, |S_n - S_{n+1}| \leq \epsilon.$$

Due to the Cauchy's test for convergence of a sequence one gets:

$$\exists S = \lim_{n \to \infty} S_n.$$

The same reasoning yields $\exists \tilde{S} = \lim_{n \to \infty} \tilde{S}_n$.

$$\forall n \, \exists N : \{\tilde{a}_k\}_{k=0}^{n} \subset \{a_m\}_{m=0}^{N}.$$

Let's consider the difference, where $m \geq N$:

$$|S - \tilde{S}_n| = |S - \tilde{S}_n + S_m - S_m| \leq |S - S_m| + |S_m - \tilde{S}_n| \to 0,$$

$$\text{as } n \to \infty \;\Rightarrow \tilde{S} = S.$$

Properties of the absolute convergent series

$$\forall c \in \mathbb{R}: \ \sum_{n=0}^{\infty} c\, a_n = c \sum_{n=0}^{\infty} a_n;$$

$$\sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} c_n, \quad c_n = (a_n + b_n);$$

$$\sum_{n=0}^{\infty} a_n \cdot \sum_{n=0}^{\infty} b_n = \sum_{k=0}^{\infty} c_k, \quad c_k = a_k \sum_{n=0}^{\infty} b_n;$$

The Dirichlet test

Let $\{a_n\}_{n=0}^{\infty}$ be a positive decreasing sequence and $a_n \to 0$ as $n \to \infty$ and the sequence $B_N = \sum_{k=0}^{N} b_n$ is bounded, then the series

$$S = \sum_{n=0}^{\infty} a_n b_n$$

converges.

Summation by parts

Let's define $B_N = \sum_{n=0}^{N} b_n$ and let B be the supremum of B_N and $b_n = B_n - B_{n-1}$, then:

$$\begin{aligned} S_N &= a_0 B_0 + a_1(B_1 - B_0) + a_2(B_2 - B_1) + \cdots + a_N(B_N - B_{N-1}) \\ &= B_0(a_0 - a_1) + B_1(a_1 - a_2) + \cdots + B_{N-1}(a_{N-1} - a_N) + \\ &+ a_N B_N = a_N B_N + \sum_{n=0}^{N-1} B_n(a_n - a_{n+1}), \end{aligned}$$

$$\sum_{n=1}^{N} a_n(B_n - B_{n-1}) = a_N B_N - a_0 b_0 - \sum_{n=0}^{N-1} B_n(a_{n+1} - a_n).$$

Abel's summation formula.

Proof of the Dirichlet test

$$\left|\sum_{n=k+1}^{k+m} a_n b_n\right| \leq \left|a_{k+m}\tilde{B}_k^{k+m} - a_k\tilde{B}_k^k - \sum_{n=k}^{k+m-1} \tilde{B}_k^n (a_{n+1} - a_n)\right|,$$

where $\tilde{B}_k^l = \sum_{n=k}^l b_n$.

$$\left|\sum_{n=k+1}^{k+m} a_n (\tilde{B}_k^n - \tilde{B}_k^{n-1})\right| \leq$$

$$\leq \left|a_{k+m}\tilde{B}_k^{k+m} - a_k\tilde{B}_k^k - \sum_{n=k}^{k+m-1} \tilde{B}_k^n (a_{n+1} - a_n)\right| \leq$$

$$\leq \left|a_{k+m}\tilde{B}_k^{k+m} - a_k\tilde{B}_k^k\right| + \left|\sum_{n=k}^{k+m-1} \tilde{B}_k^n (a_{n+1} - a_n)\right|,$$

Proof of the Dirichlet test

Define $\tilde{B} = \max_{n \geq k} |\tilde{B}_k^n|$,

$$\left| a_{k+m}\tilde{B}_k^{k+m} - a_k\tilde{B}_k^k \right| + \left| \sum_{n=k}^{k+m-1} \tilde{B}_k^n (a_{n+1} - a_n) \right| \leq$$

$$\leq 2\tilde{B}a_k + \tilde{B} \sum_{n=k}^{k+m-1} (a_n - a_{n+1}) = 2\tilde{B}a_k + \tilde{B}(a_k - a_{k+m}) \leq 3\tilde{B}a_k.$$

As a result:

$$\left| \sum_{n=k+1}^{k+m} a_n b_n \right| \leq 3\tilde{B}a_k.$$

$a_k \to 0$, $k \to \infty$, then according to the Cauchy test the series converges.

An example

$$S = \sum_{n=0}^{\infty} \frac{\sin(n)}{n+1}, \;\; a_n = \frac{1}{n+1}, \;\; b_n = \sin(n),$$

$$\sum_{n=0}^{N} \sin(n) = \sum_{n=0}^{N} \frac{\sin(n)\sin(1/2)}{\sin(1/2)} = \sum_{n=0}^{N} \frac{\cos(n-1/2) - \cos(n+1/2)}{2\sin(1/2)}$$

$$= \frac{1}{2\sin(1/2)} \left((\cos(-1/2) - \cos(1/2)) + (\cos(1/2) - \cos(3/2)) + \right.$$

$$\left. \cdots + (\cos(n-1/2) - \cos((n+1/2)) + \dots \right) =$$

$$= \frac{1}{2\sin(1/2)} (\cos(-1/2) - \cos(N+1/2)).$$

$$\left| \sum_{n=0}^{N} \sin(n) \right| = \frac{1}{2} |\cos(1/2) - \cos(N+1/2))|.$$

Estimation of a truncation error

Let's consider a convergent series:

$$S = \sum_{n=0}^{\infty} a_n.$$

Define the residual of the partial sum S_N:

$$R_N = S - S_N.$$

The truncation error is the sum of the residual terms.

Theorem about truncation error of an alternating series

An absolute value of the residue of a partial sum of an alternating series

$$S = \sum_{n=0}^{\infty} (-1)^n u_n, \; u_n > u_{n+1} > 0, \lim_{n \to \infty} u_n = 0$$

can be estimated as an absolute value of first discarded term:

$$|R_N| < |u_{n+1}|.$$

A proof of the theorem about the truncation error of an alternating series

$$R_N = \sum_{n=N+1}^{\infty} (-1)^n u_n = (-1)^{N+1}(u_{N+1} - (u_{N+2} - u_{N+3}) - (u_{N+4} - u_{N+5}) - \dots).$$

Consider

$$(-1)^{N+1} \sum_{N+1}^{\infty} (-1)^n u_n = (u_{N+1} - u_{N+2}) + (u_{N+3} - u_{N+4}) + \dots.$$

The series is the sum of positive terms, hence the partial sums increase.

A proof of the theorem about the truncation error of an alternating series

Let' rewrite the series in a following form:

$$(-1)^{N+1}\sum_{N+1}^{\infty}(-1)^n u_n = u_{N+1} - (u_{N+2} - u_{N+3}) - (u_{N+4} - u_{N+5}) + .$$

Due to decreasing the sequence $\{u_n\}$: $(u_n - n_{n+1}) > 0$ and the partial sums do not exceed u_{N+1}. Hence one can see the increasing bounded sequence of the partial sums then:

$$|R_N| < |u_{N+1}|.$$

Truncation error for absolute convergent series

Let $f(x)$ be positive decreasing function as $x > N$ for some $N > 0$ and $\exists C > 0$:

$$\int_N^\infty f(x)dx < C,$$

then the truncation error

$$R_M = \sum_{n=0}^{\infty} f(n) - \sum_{n=0}^{M} f(n),$$

$$\int_M^\infty f(x)dx < R_M < \int_{M-1}^\infty f(x)dx, \quad M > N.$$

Truncation error for absolute convergent series

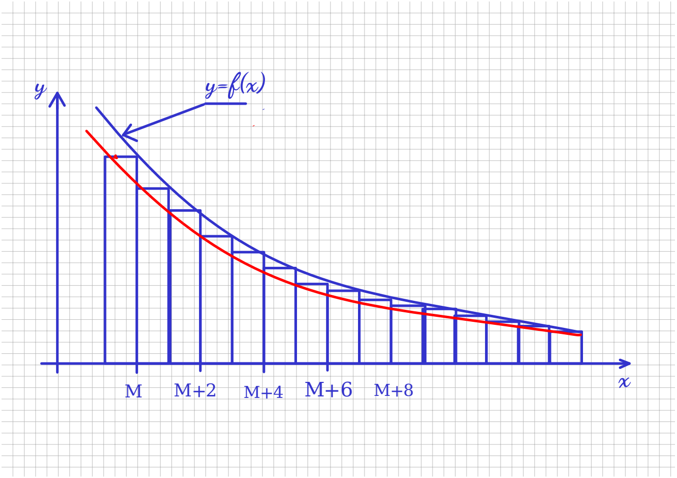

Figure: $\int_M^\infty f(x)dx < R_M < \int_{M-1}^\infty f(x)dx, \quad M > N.$

Example 1.

Using the Taylor formula for $\arctan(x)$

$$\arctan(x) \sim \sum_{n=0}^{N} \frac{(-1)^n x^{2n+1}}{2n+1}, \ \arctan(1) = \frac{\pi}{4},$$

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}, \ \frac{\pi}{4} \sim \sum_{n=0}^{10} \frac{(-1)^n x^{2n+1}}{2n+1} + R_{10},$$

$$\frac{\pi}{4} \sim 0.744011, \text{ estimation of } |R_{10}| < 1/11 \sim 0.0909,$$

$$\frac{\pi}{4} - \sum_{n=0}^{10} \frac{(-1)^n x^{2n+1}}{2n+1} \sim 0.0414$$

Example 1. An absolute convergent series for $\pi/4$

$$
\begin{aligned}
\frac{\pi}{4} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \cdots + \\
&+ \left(\frac{1}{2n+1} - \frac{1}{2n+3}\right) + \left(\frac{1}{2(n+2)+1} - \frac{1}{2(n+2)+3}\right) \cdots ; \\
&= \sum_{n=0}^{\infty} \frac{1}{4n+1} - \frac{1}{4n+3} = \sum_{n=0}^{\infty} \frac{2}{(4n+1)(4n+3)}.
\end{aligned}
$$

The last series is absolutely convergent.

Example 1. An absolute convergent series for $\pi/4$

$$\frac{\pi}{4} \sim \sum_{n=0}^{10} \frac{2}{(4n+1)(4n+3)} = 0.774040,$$

The estimation of $|R_{10}|$ yields:

$$|R_{10}| < \int_9^{\infty} \frac{2dx}{(4x+1)(4x+3)} = 0.0132,$$

$$\frac{\pi}{4} - 0.774040 \sim 0.0116.$$

Example 2. $\sqrt{2}$

Taylor formula for $\sqrt{1+x}$:

$$\sqrt{1+x} \sim 1 + \frac{x}{2} - \cdots + \frac{(-1)^{n+1}(2n)!x^n}{(2n-1)4^n(n!)^2}.$$

$$\sqrt{2} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n)!}{(2n-1)4^n(n!)^2}.$$

Example 2. $\sqrt{2}$

$$\lim_{n\to\infty} \frac{(2n)!}{(2n-1)4^n(n!)^2} = \left| n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \right| =$$

$$\lim_{n\to\infty} \frac{1}{2n-1}\frac{1}{4^n} \cdot \sqrt{2\pi(2n)}\left(\frac{2n}{e}\right)^{2n} \cdot \frac{1}{(\sqrt{2\pi n})^2}\left(\frac{e}{n}\right)^{2n} =$$

$$\lim_{n\to\infty} \frac{1}{2n-1}\frac{1}{\sqrt{\pi n}} = 0.$$

This alternating series converges.

Example 2. $\sqrt{2}$

$$\sqrt{2} \sim \sum_{n=0}^{10} \frac{(-1)^{n+1}(2n)!}{(2n-1)4^n(n!)^2} = 1.40993$$

The estimation of truncation error:

$$|R_{10}| < \frac{(2 \cdot 11)!}{(2 \cdot 11 - 1)4^{11}(11!)^2} = 0.0080,$$

$$R_{10} \sim -0.0043,$$

Example 3. Slowly convergent series

$$f(x) = \frac{1}{x\log(x)\log(\log(x))^2}, \quad S = \sum_{n=27}^{\infty} \frac{1}{n\log(n)\log(\log(n))^2}.$$

$$S_{100} = \sum_{n=27}^{100} \frac{1}{n\log(n)\log(\log(n))^2} \sim 0.1881,$$

$$\frac{1}{\log(\log(100))} = 0.6548 < R_{100} < \frac{1}{\log(\log(99))} \sim 0.6557;$$

$$S_{1000} = \sum_{n=27}^{1000} \frac{1}{n\log(n)\log(\log(n))^2} \sim 0.3250,$$

$$\frac{1}{\log(\log(1000))} < 0.5142 < R_{1000} < \frac{1}{\log(\log(999))} \sim 0.51747;$$

Summary

Alternating series

An absolute convergence

Truncation error